Exercise 1  Show that the equation
\[ x^5 + 7x^2 - 30x + 1 = 0 \]
has at least two solutions \( x_0, x_1 \in (0, 2) \). (1 p)

Exercise 2  Consider the space \( C([0, 1]) \) of all continuous maps \( f : [0, 1] \to \mathbb{R} \), endowed
with the sup-metric. Show that
\[ A := \{ f \in C([0, 1]) : x^2 \leq e^{f(x)} + \sin(f(x)) \leq x \quad \forall \ x \in [0, 1] \} \]
is a closed and bounded subset of \( C([0, 1]) \). (1 p)

Exercise 3  Describe a subspace \( X \subset \mathbb{R}^2 \) which is connected, whose closure (in \( \mathbb{R}^2 \)) is
compact, but with the property that \( X \) is not locally compact. (1 p)

Exercise 4  Let \( G = (0, \infty) \) be the group of strictly positive reals, endowed with the
usual product. Find an action of \( G \) on \( \mathbb{R}^4 \) with the property that \( \mathbb{R}^4/G \) is homeomorphic
to \( S^3 \). (1 p)

Exercise 5  Let \( X = \mathbb{R}^2 \) endowed with the product topology \( T_l \times T_l \), where \( T_l \) is the
lower limit topology on \( \mathbb{R} \).

\begin{enumerate}
  \item a. Describe a countable topology basis for the topological space \( X \). (0.5 p)
  \item b. Find a sequence \( (x_n)_{n \geq 1} \) of points in \( \mathbb{R}^2 \) which converges to \( (0, 0) \) with respect to
the Euclidean topology, but which has no convergent subsequence in the topological
space \( X \). (0.5 p)
  \item c. Compute the interior, the closure and the boundary (in \( X \)) of
\[ A = [0, 1) \times (0, 1] \]. (1p)
\end{enumerate}

(please use pictures!).

Exercise 6  Decide (and explain) which of the following statements hold true:

\begin{enumerate}
  \item a. \( S^1 \times S^1 \times S^1 \) can be embedded in \( \mathbb{R}^4 \). (0.5 p)
  \item b. \( S^1 \) can be embedded in \( (0, \infty) \). (0.5 p)
  \item c. the cylinder \( S^1 \times [0, 1] \) can be embedded in the Klein bottle. (0.5 p)
  \item d. The Moebius band can be embedded into the projective space \( \mathbb{P}^2 \). (0.5 p)
  \item e. the projective space \( \mathbb{P}^3 \) can be embedded in \( \mathbb{R}^6 \). (0.5 p)
\end{enumerate}
Exercise 7  Given a polynomial \( p \in \mathbb{R}[X_0, X_1, \ldots, X_n] \), we denote by \( \mathcal{R}_p \) the set of reminders modulo \( p \). In other words,
\[
\mathcal{R}_p = \mathbb{R}[X_0, X_1, \ldots, X_n]/R_p,
\]
where \( R_p \) is the equivalence relation on \( \mathbb{R}[X_0, X_1, \ldots, X_n] \) given by
\[
R_p = \{(q_1, q_2) : \exists q \in \mathbb{R}[X_0, X_1, \ldots, X_n] \text{ such that } q_1 - q_2 = pq\}.
\]
For \( q \in \mathbb{R}[X_0, X_1, \ldots, X_n] \), we denoted by \([q] \in \mathcal{R}_p\) the induced equivalence class. Show that:

a. The operations (on \( \mathcal{R}_p \)) +, \cdot and multiplications by scalars given by
\[
[q_1] + [q_2] := [q_1 + q_2], \quad [q_1] \cdot [q_2] := [q_1 \cdot q_2], \quad \lambda[q] := [\lambda q]
\]
are well-defined and make \( \mathcal{R}_p \) into an algebra. (0.5 p)

b. For \( p = x_0^2 + \ldots + x_n^2 \), the spectrum of \( \mathcal{R}_p \) has only one point. (0.5 p)

c. For \( p = x_0^2 + \ldots + x_n^2 - 1 \), the spectrum of \( \mathcal{R}_p \) is homeomorphic to \( S^n \) (1 p).

Note: please motivate all your answers (e.g., in Exercise 6, explain/prove in each case your answer. Or, in Exercise 4 prove that \( \mathbb{R}^4/G \) is homeomorphic to \( S^3 \)).