Exercise 1. Let $B$ be the family of subsets of $\mathbb{R}$ consisting of $\mathbb{R}$ and the subsets
\[ [n, a) := \{ r \in \mathbb{R} : n \leq r < a \} \quad \text{with} \quad n \in \mathbb{Z}, a \in \mathbb{R}. \]
1. Show that $B$ is not a topology on $\mathbb{R}$, but it is a topology basis. Denote by $T$ the associated topology. (1p)
2. Is $(\mathbb{R}, T)$ second countable? But Hausdorff? But metrizable? Can it be embedded in $\mathbb{R}^{2012}$ (with the Euclidean topology)? (1p)
3. Compute the closure, the interior and the boundary of $A = [-\frac{1}{2}, \frac{1}{2}]$ in $(\mathbb{R}, T)$. (1.5p)

Exercise 2. Prove directly that the abstract torus $T_{\text{abs}}$ is homeomorphic to $S^1 \times S^1$. More precisely, define an explicit map
\[ \tilde{f} : [0, 1] \times [0, 1] \to \mathbb{R}^4 \]
whose image is
\[ S^1 \times S^1 = \{(x, y, z, t) \in \mathbb{R}^4 : x^2 + y^2 = z^2 + t^2 = 1\} \]
and which induces a homeomorphism $f : T_{\text{abs}} \to S^1 \times S^1$. Provide all the arguments. (1.5p).

Exercise 3. Let $X$ be the space obtained from the sphere $S^2$ by gluing the north and the south pole (with the quotient topology). Show that $X$ can be obtained from a square $[0, 1] \times [0, 1]$ by glueing some of the points on the boundary (note: you are not allowed to glue a point in the interior of the square to any other point). More precisely:
1. Describe the equivalence relation $R_0$ on $S^2$ encoding the glueing that defines $X$. (0.25p)
2. Make a picture of $X$ in $\mathbb{R}^3$. (0.25p)
3. Describe an equivalence relation $R$ on $[0, 1] \times [0, 1]$ encoding a glueing with the required properties. (1p)
4. Show that, indeed, $X$ is homeomorphic to $[0, 1] \times [0, 1]/R$ (provide as many arguments as you can, but do not write down explicit maps- instead, indicate them on the picture). (0.5p)
**Exercise 4.** Show that:

1. There exist continuous surjective maps \( f : S^1 \to S^1 \) which are not injective. \((0.5p)\)

2. Any continuous injective map \( f : S^1 \to S^1 \) is surjective. \((1p)\)

**Exercise 5.** Show that any continuous map

\[ f : (\mathbb{R}, T_{\text{Eucl}}) \to (\mathbb{R}, T_l) \]

must be constant (recall that \( T_l \) is the lower limit topology- i.e. the one generated by intervals of type \([a, b)\)). \((1.5p)\)

Note 1: Motivate all your answers. Whenever you use a Theorem or Proposition, please make that clear (e.g. by stating it). Please write clearly (English or Dutch).

Note 2: The final mark is

\[ \min\{10, 1 + p\}, \]

where \( p \) is the number of points you collect from the exercises.