JUSTIFY YOUR ANSWERS

Allowed: material handed out in class and handwritten notes (your handwriting)

NOTE:

- The test consists of five problems for a total of 11.5 points
- The score is computed by adding all the points up to a maximum of 10

Problem 1. Car inspections require, on average, 45 minutes. For Monday morning a garage has scheduled an inspection at 9 AM and another at 10 AM. Both car owners come on time. Assuming that the time required by successive inspections are IID exponentially distributed random variables, determine:

(a) (1 pt.) The expected waiting time for the owner of the 10 AM car before her inspection starts.

(b) (1 pt.) The expected amount of time the owner of the 10 AM car will spend in the garage.

Answers:

(a) Let $T_1$ be the service time of the 9AM car, $T_2$ the service time of the 10AM car and $W$ the waiting time of the latter. We have

$$W = \begin{cases} 0 & \text{if } T_1 \leq 1 \\ T_1 - 1 & \text{if } T_1 > 1 \end{cases}.$$  

Hence, conditioning,

$$E(W) = E(W \mid T_1 \leq 1) P(T_1 \leq 1) + E(W \mid T_1 > 1) P(T_1 > 1)$$

$$= 0 + E(T_1 - 1 \mid T_1 - 1 > 0) e^{-60/45}$$

$$= 45 e^{-60/45} \text{ min}.$$  

The last equality is due to the memoryless property of the exponential (as shown in class).

(b) Using (a),

$$E(W + T_2) = E(W) + E(T_2) = \left[45 e^{-60/45} + 45 \right] \text{ min}.$$  

Problem 2. The arrival of customers to a shop is well approximated by a Poisson process $N(t)$ of rate $\lambda$. The $i$-th customer spends a random amount $X_i$, where the random variables $X_1, X_2, \ldots$ are independent and identically distributed with mean $\mu$ and variance $\sigma^2$. Let

$$Y(t) = \sum_{i=1}^{N(t)} X_i$$

be the revenue of the shop at time $t$.

(a) (1 pt.) Determine the mean revenue $E[Y(t)]$ for each $t > 0$.

(b) (1 pt.) Show that the variance of the revenue is $\text{Var}[Y(t)] = \lambda t (\mu^2 + \sigma^2)$.

Answers:
(a) We start by the tower property of conditional expectations:

\[ E[Y(t)] = E\left( E[Y(t) \mid N(t)] \right). \]  

We have, by the independence of \( N(t) \) and the variables \( X_i \),

\[
E[Y(t) \mid N(t) = n] = E\left[ \sum_{i=1}^{n} X_i \mid N(t) = n \right] \\
= \sum_{i=1}^{n} E[X_i \mid N(t) = n] \\
= \sum_{i=1}^{n} E[X_i] \\
= n \mu.
\]

Hence \( E[Y(t) \mid N(t)] = \mu N(t) \) and, by (??),

\[ E[Y(t)] = \mu E[N(t)] = \mu \lambda t. \]

(b) We have to compute \( E[Y(t)^2] \). We condition as in (??):

\[ E[Y(t)^2] = E\left(E[Y(t)^2 \mid N(t)]\right). \]  

By the independence of \( N(t) \) and the variables \( X_i \),

\[
E[Y(t)^2 \mid N(t) = n] = E\left[ \sum_{i,j=1}^{n} X_i X_j \mid N(t) = n \right] \\
= \sum_{i,j=1}^{n} E[X_i X_j].
\]

At this point we must distinguish the case \( i \neq j \) — involving independent variables \( X_i \) and \( X_j \)— from the case \( i = j \) for which \( X_i X_j = X_i^2 \). As there are \( n(n-1) \) pairs \((i, j)\) with \( i \neq j \), we obtain

\[
E[Y(t) \mid N(t) = n] = \sum_{i,j=1}^{n} E[X_i] E[X_j] + \sum_{i=1}^{n} E[X_i^2] \\
= n(n-1) \mu^2 + n(\sigma^2 + \mu^2) \\
= n^2 \mu^2 + n \sigma^2.
\]

Hence, \( E[Y(t)^2 \mid N(t)] = \mu^2 N(t)^2 + \sigma^2 N(t) \) and, by (??),

\[
E[Y(t)^2] = \mu^2 E[N(t)^2] + \sigma^2 E[N(t)] \\
= \mu^2 \left( \text{Var}[N(t)] + E[N(t)]^2 \right) + \sigma^2 E[N(t)] \\
= \mu^2 \left( \lambda t + (\lambda t)^2 \right) + \sigma^2 \lambda t.
\]

Using (a) we conclude

\[
\text{Var}[Y(t)] = E[Y(t)^2] - E[Y(t)]^2 = \mu^2 \lambda t + \sigma^2 \lambda t.
\]

**Problem 3.** Let \( \{N(t) : t \geq 0\} \) be a Poisson process with rate \( \lambda \). Let \( T_n \) denote the \( n \)-th inter-arrival time and \( S_n \) the time of the \( n \)-th event. Let \( t > 0 \). Find:

(a) (1 pt.) \( P(N(t) = 10, N(t/2) = 5 \mid N(t/4) = 3) \).
Consider a pure death process with three states. That is, a process whose only non-zero
rates are the death rates $\mu_1$ and $\mu_2$.

(a) Write the six non-trivial forward Kolmogorov equations.

(b) Find $P_i(t)$ for $i = 0, 1, 2$.

Answers:

(a) By independence of $N(t) - N(t/2)$, $N(t/2) - N(t/4)$ and $N(t/4)$,

$$
P(N(t) = 10, N(t/2) = 5 | N(t/4) = 3) = \frac{(\lambda t)^{5} e^{-\lambda t} (\lambda t)^{2}}{2!} e^{-\lambda t}.
$$

(b) As $T_5$, $T_6$ and $S_4$ are independent random variables,

$$
E[S_6 | S_4 = 3] = E[S_4 + T_5 + T_6 | S_4 = 3] = E[3 + T_5 + T_6 | S_4 = 3]
$$

Now we use that, under the condition $T_1 < T_2 < T_3$, the variables $T_1$, $\Delta_2$ and $\Delta_3$ are independent and, furthermore, $T_1 \sim \text{Exp}(3\lambda)$ and $\Delta_2, \Delta_3 \sim \text{Exp}(\lambda)$. Hence,

$$
E[T_3 | T_1 < T_2 < T_3] = E_{T_1=\text{min}(T_1, T_2, T_3)}[T_1 + \Delta_3 | \Delta_2 < \Delta_3]
$$

where $\bar{E}$ is an abbreviation of $E_{T_1=\text{min}(T_1, T_2, T_3)}$. We must again transcribe the remaining expectation in terms of “good” variables, in this case $\Delta_2, \Delta_3$ and $\Delta_{32} = \Delta_3 - \Delta_2$:

$$
\bar{E}[^{\Delta_3} | ^{\Delta_2} < ^{\Delta_3}] = \bar{E}_{\Delta_2=\text{min}(\Delta_2, \Delta_3)}[\Delta_2 + \Delta_{32}]
$$

Replacing in (??),

$$
E[T_3 | T_1 < T_2 < T_3] = \frac{1}{3\lambda} + \frac{1}{2\lambda} + \frac{1}{\lambda} = \frac{11}{6\lambda}.
$$

Problem 4. Consider a pure death process with three states. That is, a process whose only non-zero
rates are the death rates $\mu_1$ and $\mu_2$.

(a) Write the six non-trivial forward Kolmogorov equations.

(b) Find $P_i(t)$ for $i = 0, 1, 2$.
The system is given by:

\[
\begin{align*}
    P'_{00}(t) &= 0 \\
    P'_{10}(t) &= \mu_1 [P_{00}(t) - P_{10}(t)] \\
    P'_{11}(t) &= -\mu_1 P_{11}(t) \\
    P'_{21}(t) &= \mu_2 [P_{11}(t) - P_{21}(t)] \\
    P'_{22}(t) &= -\mu_2 P_{22}(t) \\
    P'_{20}(t) &= \mu_2 [P_{10}(t) - P_{20}(t)]
\end{align*}
\]

(b) The corresponding initial value problems and solutions are the following:

\[
\begin{align*}
    P'_{00}(t) &= 0 \quad \Rightarrow \quad P_{00}(t) = 1 \\
    P'_{11}(t) &= -\mu_1 P_{11}(t) \quad \Rightarrow \quad P_{11}(t) = e^{-\mu_1 t} \\
    P'_{22}(t) &= -\mu_2 P_{22}(t) \quad \Rightarrow \quad P_{22}(t) = e^{-\mu_2 t}.
\end{align*}
\]

Problem 5. A public job search service has a single desk and, for security reasons admits a maximum of two persons at each time. Potential applicants arrive at a Poisson rate of 4 per hour, and the successive service times are independent exponential random variables with mean equal to 10 minutes.

(a) (0.5 pts.) Write the system as a birth-and-death process with \( S = \) number of applicants present.

(b) (1 pt.) Determine the invariant measure \((P_0, P_1, P_2)\) of this process.

(c) (0.5 pts.) Determine the average number of applicants present in the office.

(d) (0.5 pts.) What proportion of time is the clerk free to read the newspaper?

Answers:

(a) \( S = \{0, 1, 2\} \), \( \lambda_0 = \lambda_1 = 4 \text{hr}^{-1} \) and \( \mu_1 = \mu_2 = 6 \text{hr}^{-1} \).

(b) The equations to be satisfied are

\[
\begin{align*}
    4 P_0 &= 6 P_1 \\
    4 P_1 &= 6 P_2 \\
    P_0 + P_1 + P_2 &= 1
\end{align*}
\]

which imply

\[
(P_0, P_1, P_2) = \left( \frac{9}{19}, \frac{6}{19}, \frac{4}{19} \right).
\]

(c) \( P_1 + 2 P_2 = 14/19 \).

(d) \( 9/19 \), approximately 47% of the time.