Exercise 1. Let $X_i, 1 = 1, \ldots, n$ be independent normal random variables with respective means $\mu_i$ and variances $\sigma_i^2$. Consider its mean 

$$ \overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i $$

(a) (0.5 pts.) Prove that $\overline{X}$ is also normally distributed.

(b) (0.5 pts.) Determine the mean and variance of $\overline{X}$.

Answers: The moment-generating function $\Phi_{\overline{X}}(t)$ of $\overline{X}$ factorizes, due to the independence of the $X_i$, in the following way:

$$ \Phi_{\overline{X}}(t) = E(e^{t\overline{X}}) = \prod_{i=1}^{n} E(e^{tX_i/n}) = \prod_{i=1}^{n} \Phi_{X_i}(t/n) $$

where $\Phi_{X_i}$ is the moment-generator function of the variable $X_i$. As each $X_i$ is normal,

$$ \Phi_{X_i}(s) = \exp \left[ \mu_i s + \frac{\sigma_i^2 s^2}{2} \right] , $$

hence

$$ \Phi_{\overline{X}}(t) = \exp \left[ t \left( \frac{1}{n} \sum_{i=1}^{n} \mu_i \right) + \frac{t^2}{2} \left( \frac{1}{n^2} \sum_{i=1}^{n} \sigma_i^2 \right) \right] . $$

This is the moment-generating function of a normal variable with mean $\frac{1}{n} \sum_{i=1}^{n} \mu_i$ and variance $\frac{1}{n^2} \sum_{i=1}^{n} \sigma_i^2$. This identifies $\overline{X}$ as a variable with such a law.

Exercise 2. (1 pt.) Consider a branching process with offspring number with mean $\mu$ and variance $\sigma$. That means, a sequence of random variables $(X_n)_{n \geq 0}$ with $X_0 = 1$ and

$$ X_n = \sum_{i=1}^{X_{n-1}} Z_i \quad n \geq 1 $$

where $Z_n$ are iid random variables (offspring distribution) independent of the $(X_n)$ with mean $\mu$. Show that $E(X_n) = \mu^n$. [Hint: Start by showing that $E(X_n) = \mu E(X_{n-1})$.]

Answer: Start with

$$ E(X_n) = E(E(X_n \mid X_{n-1}) ). $$
Now
\[ E(X_n \mid X_{n-1} = x_{n-1}) = E \left( \sum_{i \geq 1} Z_i \mid X_{n-1} = x_{n-1} \right) \]
\[ = \sum_{n \geq 1} E(Z_i \mid X_{n-1} = x_{n-1}) \]
\[ = \sum_{n \geq 1} E(Z_i) \quad \text{(independence of } Z_i \text{ and } X_{n-1}) \]
\[ = x_{n-1} \mu. \]

Hence \( E(X_n \mid X_{n-1}) = \mu X_{n-1} \) and
\[ E(X_n) = E(\mu X_{n-1}) = \mu E(X_{n-1}). \]

By induction in \( n \) we get the proposed result.

Exercise 3.
(a) (0.8 pts.) Show that
\[ \left( \begin{array}{cc} p & 1-p \\ 1-p & p \end{array} \right)^n = \left( \begin{array}{cc} 1/2 + a^n/2 & 1/2 - a^n/2 \\ 1/2 - a^n/2 & 1/2 + a^n/2 \end{array} \right) \]
for \( n \geq 1 \). Determine \( a \).

Answer: This is an easy proof by induction. Comparing for the case \( n = 1 \) we obtain \( a = 2p - 1 \).

(b) A communication system transmits the digits 0 and 1. Each digit must pass through \( n \) stages, each of which independently transmits the digit correctly with probability \( p \).

-i- (0.8 pts.) Find the probability that the final digit, \( X_n \), is correct.

Answer: \( P_{00}^n = P_{11}^n = 1/2 - (2p - 1)^n/2. \)

ii- (0.8 pts.) Find the probability that all the first \( n \) stages transmit correctly.

Answer: By independence the probability is equal to \( p^n. \)

Exercise 4. Consider a three-state Markov process \((X_n)_{n \geq 0}\) with two absorbing states. That is, a process with a three-symbol alphabet (=state space), say \( \{0, 1, 2\} \), and transition matrix

\[ P = \left( \begin{array}{ccc} 1 & 0 & 0 \\ a & b & c \\ 0 & 0 & 1 \end{array} \right) \]

with \( a, b, c > 0 \) and \( a + b + c = 1. \)

(a) (0.8 pts.) Show that \( P_{11}^n = b^n. \)

Answer: As \( P_{x1} = 0 \) for every \( x \neq 1, \)
\[ P_{11}^n = \sum_{x=0}^{2} P_{1x}^{n-1} P_{x1} = P_{11}^{n-1} P_{11} = P_{11}^{n-1} b. \]

The result follows by induction.
(b) (0.8 pts.) Show that the state “1” is transient.

**Answer:**

\[ \sum_{n \geq 0} P_{11}^n = \sum_{n \geq 0} b^n = \frac{1}{1 - b} < \infty. \]

(c) (0.8 pts.) Let \( T = \inf\{n > 0 : X_n = 0 \text{ or } X_n = 2\} \) be the time it takes the process to be absorbed in one of the absorbing states. Compute \( E(T \mid X_0 = 1). \) [Hint: you may want to use that for a discrete random variable \( Z, E[Z] = \sum_{k \geq 0} P(Z > k).\)]

**Answer:**

\[ E(T \mid X_0 = 1) = \sum_{k \geq 0} P(T > k \mid X_0 = 1) = \sum_{k \geq 0} P(X_n = 1, n = 1, \ldots, k \mid X_0 = 1) = \sum_{k \geq 0} b^k = \frac{1}{1 - b}. \]

(d) (0.8 pts.) Let \( T_0 = \inf\{n > 0 : X_n = 0\} \) and \( T_2 = \inf\{n > 0 : X_n = 2\} \) be the absorption times at each of the absorbing states. Compute \( P(T_0 < T_2 \mid X_0 = 1). \)

**Answer:**

\[ P(T_0 < T_2 \mid X_0 = 1) = \sum_{k \geq 0} P(X_{k+1} = 0, X_n = 1, n = 1, \ldots, k \mid X_0 = 0) = \sum_{k \geq 0} P(X_{k+1} = 0 \mid X_k = 1) P_{11}^k = \sum_{k \geq 0} a b^k = \frac{a}{1 - b}. \]

(e) (0.8 pts.) Compute all the invariant measures of the process.

**Answer:** Let \( \pi = (\pi_0, \pi_1, \pi_2) \) be the invariant measure. The conditions \( \sum \pi_x P_{xy} = \pi_y \) plus the normalisation condition \( \pi_0 + \pi_1 + \pi_2 = 1 \) become:

\[
\begin{align*}
\pi_0 + a \pi_1 &= \pi_0 \\
\pi_1 &= b \pi_1 \\
c \pi_1 + \pi_2 &= \pi_2 \\
\pi_0 + \pi_1 + \pi_2 &= 1.
\end{align*}
\]

All their solutions are of the form \( \pi_0 = 0, \pi_1 + \pi_2 = 1. \) That is, the invariant measures \( \pi \) take the form

\[
\pi = (\lambda, 0, 1 - \lambda) = \lambda (1, 0, 0) + (1 - \lambda) (0, 0, 1) \quad \text{for } 0 \leq \lambda \leq 1,
\]

That is, the invariant measures are convex combinations of the measure concentrated in the state “0” and the measure concentrated in the state “2”.

**Exercise 5.** At a certain beach resort a bad day is equally likely to be followed by a good or a bad day, while a good day is five times more likely to be followed by a good day than by a bad day. The number of interventions by lifesavers is Poisson distributed with mean 4 in good days and mean 1 in bad days. Find, in the long run,

(a) (0.8 pts.) The probability of the lifesavers not having any intervention in a given day.

(b) (0.8 pts.) The average number of interventions per day.
[Take $e^{-4} \sim 0.02$ and $e^{-1} \sim 0.4$.]

**Answers:** Associating “good days” → 1 and “bad days” → 2, the weather pattern is a Markov process with transition matrix

$$
\begin{pmatrix}
5/6 & 1/6 \\
1/2 & 1/2
\end{pmatrix}
$$

The proportion of good and bad days is determined, in the long run, by the invariant measure $\Pi$ of this chain. This measure satisfies:

$$
\begin{align*}
\frac{5}{6}\Pi_1 + \frac{1}{2}\Pi_2 &= \Pi_1 \\
\Pi_1 + \Pi_2 &= 1
\end{align*}
\implies \Pi = \left( \frac{3}{4}, \frac{1}{4} \right).
$$

(a) Let $S$ be the number of interventions per day.

$$
P(S = 0) = P(S = 0 \mid \text{good day}) P(\text{good day}) + P(S = 0 \mid \text{bad day}) P(\text{bad day})
= e^{-4} \frac{3}{4} + e^{-1} \frac{1}{4} \sim 0.11
$$

(b)

$$
E(S) = E(S \mid \text{good day}) P(\text{good day}) + E(S \mid \text{bad day}) P(\text{bad day})
= 4 \cdot \frac{3}{4} + 1 \cdot \frac{1}{4} = \frac{13}{4} = 3.25.
$$

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**Bonus problem**

**Bonus** Consider a homogeneous (or shift-invariant) Markov chain $(X_n)_{n \in \mathbb{N}}$ with finite state space $S$. Let us recall that the hitting time of a state $y$ is

$$T_y = \min \{n \geq 1 : X_n = y\}.$$

(a) If $\ell \leq n \in \mathbb{N}$, $x, y \in S$, prove the following

- i- (0.5 pts.)

$$P(X_n = y, T_y = \ell \mid X_0 = x) = P^n_{yy} P(T_y = \ell \mid X_0 = x).$$

**Answer:** Decomposing in terms of trajectories,

$$P(X_n = y, T_y = \ell \mid X_0 = x) = \sum_{x_1, \ldots, x_{\ell-1} \neq y} P(X_n = y, X_{\ell} = y, X_{\ell-1} = x_{\ell-1}, \ldots, X_1 = x_1 \mid X_0 = x)$$

$$= \sum_{x_1, \ldots, x_{\ell-1} \neq y} P^n_{yy} P(X_{\ell} = y, X_{\ell-1} = x_{\ell-1}, \ldots, X_1 = x_1 \mid X_0 = x)$$

$$= \sum_{x_1, \ldots, x_{\ell-1} \neq y} P^n_{yy} P(T_y = \ell \mid X_0 = x)$$

- ii- (0.5 pts.)

$$P^n_{xy} = \sum_{\ell=1}^{n} P^n_{yy} P(T_y = \ell \mid X_0 = x).$$
Answer: As
\[ \{ X_n = y \} = \bigcup_{\ell=1}^{n} \{ X_n = y, T_y = \ell \} , \]
the union being disjoint, we conclude that
\[ \sum_{\ell=1}^{n} P(X_n = y, T_y = \ell \mid X_0 = x) = P(X_n = y \mid X_0 = x) = P^n_{xy} . \]
The result follows, hence, by summing both sides of -i- with respect to \( \ell \).

(b) Conclude the following:

-i- (0.5 pts.) If every state is transient, then for every \( x, y \in S \).
\[ \sum_{n \geq 0} P^n_{xy} < \infty . \]

Answer: By -ii- above,
\[ \sum_{n \geq 0} P^n_{xy} = \sum_{n \geq 0} \sum_{\ell=1}^{n} P^n_{yy} P(T_y = \ell \mid X_0 = x) . \]
Hence, interchanging the order of summation,
\[ \sum_{n} P^n_{xy} = \sum_{\ell \geq 1} \sum_{n \geq \ell} P^n_{yy} P(T_y = \ell \mid X_0 = x) = \sum_{\ell \geq 1} P(T_y = \ell \mid X_0 = x) \sum_{m \geq 0} P^m_{yy} = P(T_y < \infty \mid X_0 = x) \sum_{m \geq 0} P^m_{yy} . \]
If \( y \) is transient, the last sum is finite.

-ii- (0.5 pts.) The previous result leads to a contradiction with the stochasticity property of the matrix \( \mathbb{P} \). Hence not all states can be transient.
Answer: Summing over \( y \) the inequality in (b)-i- we get
\[ \sum_{y} \sum_{n \geq 0} P^n_{xy} < \infty \quad (1) \]
(recall that \( S \) is finite). However, by stochasticity \( \sum_{y} P^n_{xy} = 1 \) for every \( n \geq 0 \). Hence,
\[ \sum_{y} \sum_{n \geq 0} P^n_{xy} = \sum_{n \geq 0} \sum_{y} P^n_{xy} = \sum_{n \geq 0} 1 = \infty , \]
in contradiction with (1).