Differentiable manifolds – Exam 1

Notes:
1. Write your name and student number **clearly** on each page of written solutions you hand in.
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are not allowed to consult any text book, class notes, colleagues, calculators, computers etc.
5. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

Some definitions you should know, but may have forgotten.

- Given a smooth map \( f : M \rightarrow N \), the **critical points** of \( f \) are the points \( p \in M \) where \( f_\ast|_p : T_pM \rightarrow T_{f(p)}N \) is not surjective. The **critical values** of \( f \) are the points in \( N \) which are images of critical points.

- The *quaternions*, \( \mathbb{H} \), are isomorphic to \( \mathbb{R}^4 \) as vector space and are endowed with a multiplication which makes \( \mathbb{H}\{0\} \) into a group. This multiplication is \( \mathbb{R} \)-bilinear and is defined on a basis \( \{1, i, j, k\} \) of \( \mathbb{H} \) by
  \[
i^2 = j^2 = k^2 = ijk = -1.\]

- A *Lie group* is a differentiable manifold \( G \) endowed with the structure of a group and for which the maps
  \[
  G \times G \rightarrow G \quad (g, h) \mapsto g \cdot h
  
  G \rightarrow G \quad g \mapsto g^{-1}
  \]
  are smooth.

Questions

1) Show that the sphere

\[
S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\} \subset \mathbb{R}^{n+1}
\]

is a manifold.

2) Let \( M \) be the subset of \( \mathbb{R}^3 \) defined by the equation

\[
M = \{(x_1, x_2, x_3) : x_1^3 + x_2^3 + x_3^3 = 1\}.
\]
a) Show that $M$ is a smooth submanifold of $\mathbb{R}^3$;

b) Define $\pi: M \longrightarrow \mathbb{R}; \pi(x_1, x_2, x_3) = x_1$. Find the critical points and critical values of $\pi$.

3) Let $M$ be a compact manifold of dimension $n$ and let $f: M \longrightarrow \mathbb{R}^n$ be smooth. Show that $f$ has a critical point.

4a) Show that $\mathbb{H}\{0\}$ is a Lie group if endowed with quaternionic multiplication as group operation.

4b) Show that the 3-dimensional sphere $S^3 \subset \mathbb{H}\{0\}$ is also a Lie group with quaternionic multiplication as group operation.

5) Given a smooth map $\varphi: M \longrightarrow N$ it induces pullback maps on 0- and 1-forms, all of them denoted by $\varphi^*$, defined by

$$
\varphi^*: \Omega^0(N) \longrightarrow \Omega^0(M) \quad \varphi^* f = f \circ \varphi;
$$

$$
\varphi^*: \Omega^1(N) \longrightarrow \Omega^1(M) \quad \alpha \mapsto \varphi^* \alpha;
$$

$$
\varphi^* \alpha|_p(X) = \alpha|_{\varphi(p)}(\varphi_* X) \quad \forall X \in T_p M.
$$

Show that if $f \in \Omega^0(N)$, then $\varphi^* df = d(\varphi^* f)$. 