Geometry and Topology – Exam 1

Notes:
1. Write your name and student number clearly on each page of written solutions you hand in.
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are allowed to consult text books and class notes.
5. You are not allowed to consult colleagues, calculators, computers etc.
6. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

Questions

Exercise 1.
1. Show that if \( X = X_1 \cup X_2 \) is a CW complex, \( X_1, X_2 \) are subcomplexes and \( X_1, X_2 \) and \( X_1 \cap X_2 \) are contractible then \( X \) is contractible.
2. Show that \( X \) is contractible if and only if all maps \( f : X \to Y \) are null homotopic for every topological space \( Y \).

Exercise 2. For \( g, n \) positive integers, let \( \Sigma_{g,n} \) be surface obtained by removing \( n \) points from the connected sum of \( g \) tori. Compute \( \pi_1(\Sigma_{g,n}) \). Compute the Abelianization \( \pi_1(\Sigma_{g,n})/[\pi_1(\Sigma_{g,n})] \).

Exercise 3.
1. Show that if a space \( X \) is obtained from a path connected space \( X_0 \) by attaching \( n \)-cells with \( n > 2 \) then the inclusion \( \iota : X_0 \hookrightarrow X \) induces an isomorphism of fundamental groups: \( \iota_* : \pi_1(X_0, x_0) \xrightarrow{\cong} \pi_1(X, x_0) \).
2. Using the previous result or otherwise, compute \( \pi_1(\mathbb{R}P^n) \) for \( n > 1 \) and \( \pi_1(\mathbb{C}P^n) \) for \( n > 0 \).

Exercise 4. A semigroup is a set \( X \) endowed with a map \( m : X \times X \to X \) and an element \( e \in X \) such that \( m(e, x) = m(x, e) = x \) for all \( x \in X \). A topological semigroup is topological space which is a semigroup and for which the multiplication \( m \) is continuous. Following the steps below or otherwise prove that if \( X \) is a topological semigroup, then \( \pi_1(X, e) \) is an Abelian group.

- Define an operation on loops based at \( e \) by
  \[
  \gamma_1 \star \gamma_2(t) := m(\gamma_1(t), \gamma_2(t)), \quad \text{for all } \gamma_i : (I, \partial I) \to (X, e).
  \]
  Show that if \( \gamma'_i \) is homotopic to \( \gamma_i \) as loops based at \( e \) then \( \gamma_1 \star \gamma_2 \) is homotopic to \( \gamma'_1 \star \gamma'_2 \). Conclude that \( \star \) defines an operation on \( \pi_1(X, e) \):
  \[
  \star : \pi_1(X, e) \times \pi_1(X, e) \to \pi_1(X, e).
  \]
• Letting $\cdot$ denote concatenation of paths and $e$ denote the constant loop, use that $\gamma_1 \simeq \gamma_1 \cdot e$ and $
abla 2 \simeq e \cdot \gamma_2$ to conclude that $\ast$ agrees with the usual product on $\pi_1(X,e)$.

• Using that $\gamma_1 \simeq e \cdot \gamma_1$ and $\gamma_2 \simeq \gamma_2 \cdot e$, conclude that $\pi_1(X,e)$ is Abelian.

Exercise 5. Let $p : \tilde{X} \to X$ be a path connected and simply connected covering of $X$ and let $A \subset X$ be a path connected and locally path connected subset. Let $\tilde{A} \subset \tilde{X}$ be a path connected component of $p^{-1}(A)$. Show that $p|_{\tilde{A}} : \tilde{A} \to A$ is a covering map corresponding to the kernel of the map $\iota_* : \pi_1(A) \to \pi_1(X)$. 