(1) Let \((X, \mathcal{B}, \nu)\) be a measure space, and suppose \(X = \bigcup_{n=1}^{\infty} E_n\), where \(\{E_n\}\) is a collection of pairwise disjoint measurable sets such that \(\nu(E_n) < \infty\) for all \(n \geq 1\). Define \(\mu\) on \(\mathcal{B}\) by \(\mu(B) = \sum_{n=1}^{\infty} 2^{-n} \nu(B \cap E_n)/(\nu(E_n) + 1)\).

(a) Prove that \(\mu\) is a finite measure on \((X, \mathcal{B})\). (10 pt.)

(b) Let \(B \in \mathcal{B}\). Prove that \(\mu(B) = 0\) if and only if \(\nu(B) = 0\). (10 pt.)

**Proof (a):** Clearly \(\mu(\emptyset) = 0\), and

\[
\mu(X) = \sum_{n=1}^{\infty} 2^{-n} \nu(E_n)/(\nu(E_n) + 1) \leq \sum_{n=1}^{\infty} 2^{-n} = 1 < \infty.
\]

Now, let \((C_n)\) be a disjoint sequence in \(\mathcal{B}\). Then,

\[
\mu(\bigcup_{m=1}^{\infty} C_m) = \sum_{n=1}^{\infty} 2^{-n} \nu((\bigcup_{m=1}^{\infty} C_m) \cap E_n)/(\nu(E_n) + 1)
= \sum_{n=1}^{\infty} 2^{-n} \sum_{m=1}^{\infty} \nu(C_m \cap E_n)/(\nu(E_n) + 1)
= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} 2^{-n} \nu(C_m \cap E_n)/(\nu(E_n) + 1)
= \sum_{m=1}^{\infty} \mu(C_m).
\]

Thus, \(\mu\) is a finite measure.

**Proof (b):** Suppose that \(\nu(B) = 0\), then \(\nu(B \cap E_n) = 0\) for all \(n\), hence \(\mu(B) = 0\). Conversely, suppose \(\mu(B) = 0\), then \(\nu(B \cap E_n) = 0\) for all \(n\). Since \(X = \bigcup_{n=1}^{\infty} E_n\) (disjoint union), then

\[
\nu(B) = \nu(B \cap \bigcup_{n=1}^{\infty} E_n) = \nu(\bigcup_{n=1}^{\infty} (B \cap E_n)) = \sum_{n=1}^{\infty} \nu(B \cap E_n) = 0.
\]

(2) Consider the measure space \((\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)\), where \(\mathcal{B}(\mathbb{R})\) is the Borel \(\sigma\)-algebra, and \(\lambda\) Lebesgue measure. Determine the value of \(\lim_{n \to \infty} \int_{(0,n)} x^2 \left( 1 - \frac{x}{n} \right)^n d\lambda(x)\). (20 pt.)

**Proof:** Let \(u_n(x) = 1_{(0,n)} x^2 \left( 1 - \frac{x}{n} \right)^n\), then \(\lim_{n \to \infty} u_n(x) = 1_{(0,\infty)} x^2 e^{-x}\). Using the fact that \((1 - \frac{x}{n})^n \to e^{-x}\), we see that \(u_n(x) \leq 1_{(0,\infty)} x^2 e^{-x}\). Since the function \(x^2 e^{-x}\) is measurable, non-negative and the improper Riemann integrable on \([0,\infty)\) exists, it follows that it is Lebesgue integrable on \([0,\infty)\) (and hence also on \((0,\infty)\)) and its value equals the improper Riemann integral. By Lebesgue Dominated Convergence Theorem, we have

\[
\lim_{n \to \infty} \int_{(0,n)} x^2 \left( 1 - \frac{x}{n} \right)^n d\lambda(x) = \lim_{n \to \infty} \int u_n(x) d\lambda(x)
= \int 1_{(0,\infty)} x^2 e^{-x} d\lambda(x) = \int_{0}^{\infty} x^2 e^{-x} dx = 2.
\]
(3) Let \( X \) be a set, and \( C \subseteq \mathcal{P}(X) \). Consider \( \sigma(C) \), the smallest \( \sigma \)-algebra over \( X \) containing \( C \), and let \( D \) be the collection of sets \( A \in \sigma(C) \) with the property that there exists a countable collection \( C_0 \subseteq C \) (depending on \( A \)) such that \( A \in \sigma(C_0) \).

(a) Show that \( D \) is a \( \sigma \)-algebra over \( X \). (12 pt.)
(b) Show that \( \mathcal{D} = \sigma(C) \). (8 pt.)

**Proof (a):** Clearly \( \emptyset \in D \) since \( \emptyset \) belongs to every \( \sigma \)-algebra. Let \( A \in D \), then there is a countable collection \( C_0 \subseteq C \) such that \( A \in \sigma(C_0) \). But then \( A^c \in \sigma(C_0) \), hence \( A^c \in D \). Finally, let \( \{ A_n \} \) be in \( D \), then for each \( n \) there exists a countable collection \( C_n \subseteq C \) such that \( A_n \in \sigma(C_n) \). Let \( C_0 = \bigcup_n C_n \), then \( C_0 \subseteq C \), and \( C_0 \) is countable. Furthermore, \( \sigma(C_n) \subseteq \sigma(C_0) \), and hence \( A_n \in \sigma(C_0) \) for each \( n \) which implies that \( \bigcup_n A_n \in \sigma(C_0) \). Therefore, \( \bigcup_n A_n \in D \) and \( D \) is a \( \sigma \)-algebra.

**Proof (b):** By definition \( D \subseteq \sigma(C) \). Also, \( C \subseteq D \) since \( C \in \sigma(\{ C \}) \) for every \( C \in C \). Since \( \sigma(C) \) is the smallest \( \sigma \)-algebra over \( X \) containing \( C \), then by part (a) \( \sigma(C) \subseteq D \). Thus, \( D = \sigma(C) \).

(4) Let \((X,A,\mu_1)\) and \((Y,B,\nu_1)\) be \(\sigma\)-finite measure spaces. Suppose \(f \in L^1(\mu_1)\) and \(g \in L^1(\nu_1)\) are non-negative. Define measures \(\mu_2\) on \(A\) and \(\nu_2\) on \(B\) by

\[
\mu_2(A) = \int_A f \, d\mu_1 \quad \text{and} \quad \nu_2(B) = \int_B g \, d\nu_1,
\]

for \(A \in A\) and \(B \in B\).

(a) For \(D \in A \otimes B\) and \(y \in Y\), let \(D_y = \{ x \in X : (x,y) \in D \}\). Show that if \(\mu_1(D_y) = 0 \nu_1\) a.e., then \(\mu_2(D_y) = 0 \nu_2\) a.e. (7 pt.)

(b) Show that if \(D \in A \otimes B\) is such that \((\mu_1 \times \nu_1)(D) = 0\) then \((\mu_2 \times \nu_2)(D) = 0\). (6 pt.)

(c) Show that for every \(D \in A \otimes B\) one has

\[
(\mu_2 \times \nu_2)(D) = \int_D f(x)g(y) \, d(\mu_1 \times \nu_1)(x,y).
\]

(7 pt.)

**Proof (a)** Suppose \(\mu_1(D_y) = 0 \nu_1\) a.e. Let \(B = \{ y \in Y : \mu_1(D_y) > 0 \}\), and \(C = \{ y \in Y : \mu_2(D_y) > 0 \}\). By our assumption, \(\nu_1(B) = 0\). By Theorem 10.9(ii), for any \(y \in Y \setminus B\) one has \(\mu_2(D_y) = 0\). Thus, \(C \subseteq B\), so that \(\nu_1(C) = 0\). Applying Theorem 10.9(ii) again, we see that \(\nu_2(C) = 0\). Thus, \(\mu_2(D_y) = 0 \nu_2\) a.e.

**Proof (b)** Suppose that \(D \in A \otimes B\) is such that \((\mu_1 \times \nu_1)(D) = 0\). Then,

\[
\int \mu_1(D_y) \, d\nu_1(y) = (\mu_1 \times \nu_1)(D) = 0.
\]

By Theorem 10.9(i), we have that \(\mu_1(D_y) = 0 \nu_1\) a.e. By part (a) above this implies that \(\mu_2(D_y) = 0 \nu_2\) a.e. Thus, by Theorem 10.9(i)

\[
(\mu_2 \times \nu_2)(D) = \int \mu_2(D_y) \, d\nu_2(y) = 0.
\]

**Proof (c)** By Tonelli’s Theorem, we have

\[
(\mu_2 \times \nu_2)(D) = \int_Y \int_X 1_{D_y}(x) \, d\mu_2(x) \, d\nu_2(y) = \int_Y \left( \int_X 1_{D_y}(x) \, f(x) \, d\mu_1(x) \right) \, d\nu_2(y) = \int_Y \left( \int_X 1_{D_y}(x) \, f(x) \, d\mu_1(x) \right) \, g(y) \, d\nu_1(y) = \int_Y \int_X 1_D(x,y) \, f(x) \, g(y) \, d\mu_1(x) \, d\nu_1(y) = \int_X \int_Y 1_D(x,y) \, f(x) \, g(y) \, d(\mu_1 \times \nu_1)(x,y) = \int_D f(x)g(y) \, d(\mu_1 \times \nu_1)(x,y).
\]
(5) Let \((X, A, \mu)\) be a probability space and let \(f \in \mathcal{M}(A)\). Suppose \((f_n) \subset \mathcal{M}(A)\) converges in measure to \(f\), i.e. \(f_n \xrightarrow{\mu} f\).

(a) Show that there exists a sequence \(n_1 < n_2 < \cdots\) such that
\[
\mu(\{x \in X : |f_{n_k}(x) - f(x)| > 1/k\}) \leq 2^{-k},
\]
for all \(k \geq 1\). (8 pt.)

(b) Let \(A = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k\). Show that \(\mu(A) = 0\), and
\[
\lim_{n \to \infty} f_{n_k}(x) = f(x) \quad \text{for all } x \not\in A.
\]
Conclude that \(f_n \to f \mu\text{-a.e.} \) (12 pt.)

**Proof(a)** Using convergence in measure, the sequence \(n_k\) is defined inductively as follows. Starting with \(\epsilon_1 = 1\), we find \(n_1\) such that
\[
\mu(\{x \in X : |f_{n_1}(x) - f(x)| > 1\}) \leq 2^{-1}.
\]
Now choose \(\epsilon_2 = \frac{1}{2}\), we find \(n_2 > n_1\) such that
\[
\mu(\{x \in X : |f_{n_2}(x) - f(x)| > 1/2\}) \leq 2^{-2}.
\]
Continuing in this manner, we find at the \(k\)th stage an \(n_k > n_{k-1}\) such that
\[
\mu(\{x \in X : |f_{n_k}(x) - f(x)| > 1/k\}) \leq 2^{-k}.
\]

**Proof(b)** Let \(A_k = \{x \in X : |f_{n_k}(x) - f(x)| > 1/k\}\) and \(A = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k\). By part (a)
\[
\mu(A_k) \leq 2^{-k}\quad \text{and hence } \sum_{k=1}^{\infty} \mu(A_k) < \infty.
\]
By Borel-Cantelli Lemma (Exercise 6.9), we have \(\mu(A) = 0\). For \(x \not\in A\), there exists \(n \geq 1\) such that \(x \not\in \bigcup_{k=n}^{\infty} A_k\). This implies that \(x \not\in A_k\) for all \(k \geq n\) and therefore \(|f_{n_k}(x) - f(x)| \leq 1/k\) for all \(k \geq n\). Thus,
\[
\lim_{k \to \infty} f_{n_k}(x) = f(x) \quad \text{for all } x \not\in A.\]
Since \(\mu(X \setminus A) = 1\) we have that \(f_{n_k} \to f \mu\text{-a.e.} \)