Measure and Integration: Solutions Quiz 2012-13

1. Consider the measure space \(([0,1), \mathcal{B}([0,1)), \lambda)\), where \(\mathcal{B}([0,1))\) is the Borel \(\sigma\)-algebra restricted to \([0,1)\) and \(\lambda\) is the restriction of Lebesgue measure on \([0,1)\). Define the transformation \(T : [0,1) \to [0,1)\) given by

\[
T(x) = \begin{cases} 
3x & 0 \leq x < 1/3, \\
3x - 1 & 1/3 \leq x < 2/3 \\
3x - 2 & 2/3 \leq x < 1.
\end{cases}
\]

(a) Show that \(T\) is \(\mathcal{B}([0,1))/\mathcal{B}([0,1))\) measurable.

(b) Determine the image measure \(T(\lambda) = \lambda \circ T^{-1}\).

(c) Let \(C = \{A \in \mathcal{B}([0,1)) : T^{-1}A = A\}\). Show that \(C\) is a \(\sigma\)-algebra.

**Solution (a):** To show \(T\) is \(\mathcal{B}([0,1))/\mathcal{B}([0,1))\) measurable, it is enough to consider inverse images of intervals of the form \([a, b) \subset [0,1)\). Now,

\[
T^{-1}([a, b)) = [\frac{a}{3}, \frac{b}{3}) \cup [\frac{a+1}{3}, \frac{b+1}{3}) \cup [\frac{a+2}{3}, \frac{b+2}{3}) \in \mathcal{B}([0,1)).
\]

Thus, \(T\) is measurable.

**Solution (b):** We claim that \(T(\lambda) = \lambda\). To prove this, we use Theorem 5.7. Notice that \(\mathcal{B}([0,1))\) is generated by the collection \(\mathcal{G} = \{[a, b) : 0 \leq a \leq b < 1\}\) which is closed under finite intersections. Now,

\[
T(\lambda)([a, b)) = \lambda(T^{-1}([a, b))) = \lambda([\frac{a}{3}, \frac{b}{3}) \cup [\frac{a+1}{3}, \frac{b+1}{3}) \cup [\frac{a+2}{3}, \frac{b+2}{3})) = b - a = \lambda([a, b)).
\]

Since the constant sequence \(([0,1))\) is exhausting, belongs to \(\mathcal{G}\) and \(\lambda([0,1)) = T(\lambda([0,1)) = 1 < \infty\), we have by Theorem 5.7 that \(T(\lambda) = \lambda\).

**Solution (c):** We check the three conditions for a collection of sets to be a \(\sigma\)-algebra. Firstly, the empty set \(\emptyset \in \mathcal{B}([0,1))\) and \(T^{-1}(\emptyset) = \emptyset\), hence \(\emptyset \in C\). Secondly, Let \(A \in C\), then \(T^{-1}A = A\). Now,

\[
T^{-1}(X \setminus A) = T^{-1}X \setminus T^{-1}A = X \setminus T^{-1}A = X \setminus A.
\]
Thus, $X \setminus A \in \mathcal{B}([0, 1))$ and $T^{-1}(X \setminus A) = X \setminus A$. This implies $X \setminus A \in \mathcal{C}$. Thirdly, let $(A_n)$ be a sequence in $\mathcal{C}$, then $A_n \in \mathcal{B}([0, 1))$ and $T^{-1}A_n = A_n$ for each $n$. Since $\mathcal{B}([0, 1))$ is a $\sigma$-algebra, we have $\bigcup_n A_n \in \mathcal{B}([0, 1))$, and

$$T^{-1}\left(\bigcup_n A_n\right) = \bigcup_n T^{-1}A_n = \bigcup_n A_n.$$ 

Thus, $\bigcup_n A_n \in \mathcal{C}$. This shows that $\mathcal{C}$ is a $\sigma$-algebra.

2. Let $\mathcal{B}(\mathbb{R}^n)$ be the Borel $\sigma$-algebra over $\mathbb{R}^n$, and let $\overline{\mathcal{B}(\mathbb{R}^n)}$ be the completion of $\mathcal{B}(\mathbb{R}^n)$ (In the notation of exercise 4.13, p.29, if $\mathcal{A} = \mathcal{B}(\mathbb{R}^n)$, then $\mathcal{A}^* = \overline{\mathcal{B}(\mathbb{R}^n)}$). The $\sigma$-algebra $\overline{\mathcal{B}(\mathbb{R}^n)}$ is called the Lebesgue $\sigma$-algebra over $\mathbb{R}^n$. Let $n = 1$ and suppose $M \subset \mathbb{R}$ is a non-Lebesgue measurable set (i.e. $M \notin \overline{\mathcal{B}(\mathbb{R})}$). Define $A = \{(x, x) \in \mathbb{R}^2 : x \in M\}$, and let $g : \mathbb{R} \rightarrow \mathbb{R}^2$ be given by $g(x) = (x, x)$.

(a) Show that $A \in \overline{\mathcal{B}(\mathbb{R}^2)}$ i.e. $A$ is Lebesgue measurable.

(b) Show that $g$ is a Borel measurable function, i.e. $g^{-1}(B) \in \mathcal{B}(\mathbb{R})$ for each $B \in \mathcal{B}(\mathbb{R}^2)$.

(c) Show that $A \notin \overline{\mathcal{B}(\mathbb{R}^2)}$, i.e. $A$ is not Borel measurable.

**Solution(a):** The set $A$ is a subset of the diagonal line $L = \{(x, x) : x \in \mathbb{R}\}$ which is the image of the hyperplane $x = 0$ (the y-axis) under a rotation by 45° which is a linear transformation. Hence by exercise 6.4 and Theorem 7.9 we have that $L \in \mathcal{B}(\mathbb{R}^2)$ and $\lambda^2(L) = 0$. Thus, $A$ is a subset of a Borel set of measure zero, by exercise 4.13(iv) we have $A \in \overline{\mathcal{B}(\mathbb{R}^2)}$.

**Solution(b):** This follows from the simple fact that $g$ is a continuous function and hence $\mathcal{B}(\mathbb{R})/\overline{\mathcal{B}(\mathbb{R}^2)}$ measurable (see example 7.3).

**Solution(c):** We give a proof by contradiction. Suppose that $A \in \overline{\mathcal{B}(\mathbb{R}^2)}$, since $G$ is measurable, then $g^{-1}(A) \in \mathcal{B}(\mathbb{R})$. However, $T^{-1}(A) = M$ and $M \notin \mathcal{B}(\mathbb{R})$, leading to a contradiction. Thus, $A \notin \mathcal{B}(\mathbb{R}^2)$.

3. Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where $\mathcal{B}(\mathbb{R})$ is the Borel $\sigma$-algebra over $\mathbb{R}$, and $\lambda$ is Lebesgue measure. Let $f_n : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$f_n(x) = \sum_{k=0}^{2^n-1} \frac{k}{2^n} \cdot 1_{[k/2^n,(k+1)/2^n)}(x), \ n \geq 1.$$ 

(a) Show that $f_n$ is measurable, and $f_n(x) \leq f_{n+1}(x)$ for all $x \in X$.

(b) Let $f(x) = x1_{(0,1)}(x)$. Show that $f$ is $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$ measurable.

(c) Prove that $f(x) = \lim_{n \rightarrow \infty} f_n(x) = \sup_{n \geq 1} f_n(x)$ for all $x \in \mathbb{R}$.

(d) Show that $\int f d\lambda = \frac{1}{2}$.

**Solution(a):** Since $[k/2^n, (k+1)/2^n) \in \mathcal{B}(\mathbb{R})$, then $1_{[k/2^n,(k+1)/2^n)}$ is a measurable function. Thus $f_n$ is a linear combination of measurable functions (in fact $f_n$ is a
simple function) and hence measurable. For \( x \notin [0, 1) \), we have \( f_n(x) = f_{n+1}(x) = 0 \). Suppose \( x \in [0, 1) \), then there exists a \( 0 \leq k \leq 2^n - 1 \) such that \( x \in [k/2^n, (k+1)/2^n) \). Since

\[
[k/2^n, (k+1)/2^n) = [2k/2^{n+1}, (2k+1)/2^{n+1}) \cup [(2k+1)/2^{n+1}, (2k+2)/2^{n+1}),
\]
we see that \( f_n(x) = k/2^n \) while \( f_{n+1}(x) \in \{2k/2^{n+1}, 2k+1/2^{n+1}\} \) so that \( f_n(x) \leq f_{n+1}(x) \).

**Solution(b):** We consider inverse images of interval of the form \([a, \infty)\). Now,

\[
f^{-1}([a, \infty)) = \begin{cases} 
\mathbb{R} & a \leq 0, \\
[a, 1) & 0 < a < 1 \\
\emptyset & a \geq 1.
\end{cases}
\]

In all cases we see that \( f^{-1}([a, \infty)) \in \mathcal{B}(\mathbb{R}) \). Thus, \( f \) is \( \mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R}) \) measurable.

**Solution(c):** For \( x \notin [0, 1) \), we have \( f(x) = f_n(x) = 0 \) for all \( n \). For \( x \in [0, 1) \), there exists for each \( n \), an integer \( k_n \in \{0, 1, \ldots, 2^n-1\} \) such that \( x \in [k_n/2^n, (k_n+1)/2^n) \). Thus,

\[
|x - k_n/2^n| = |f(x) - f_n(x)| < \frac{1}{2^n}.
\]
Since \( f_n \) is an increasing sequence, we have

\[
f(x) = \lim_{n \to \infty} f_n(x) = \sup_n f_n(x).
\]

**Solution(d):** We apply Beppo-Levi,

\[
\int f \, d\lambda = \lim_{n \to \infty} \int f_n \, d\lambda = \lim_{n \to \infty} \sum_{k=0}^{2^n-1} k \frac{1}{2^n} \lambda([k/2^n, (k+1)/2^n)) = \frac{1}{2^n} \sum_{k=0}^{2^n-1} k = \frac{1}{2^n} \cdot \frac{(2^n - 1)2^n}{2} = \frac{1}{2}.
\]

4. Let \((X, \mathcal{A}, \mu)\) be a measure space, and \( u \in \mathcal{M}_+^*(\mathcal{A}) \) satisfying \( \int u \, d\mu < \infty \). For \( a > 0 \) (a real number) set \( B_a = \{x \in X : u(x) > a\} \).

(a) Show that for any \( a > 0 \) one has

\[
a \mathbf{1}_{B_a}(x) \leq u(x) \quad \text{for all } x \in X.
\]

(b) Prove that \( \mu(B_a) < \infty \) for all \( a > 0 \).
(c) Assume that \( u(x) > 0 \) for all \( x \in X \), i.e. \( u \) is strictly positive. Show that \( \mu \) is \( \sigma \)-finite, i.e. there exists an exhausting sequence \( A_n \uparrow X \) with \( \mu(A_n) < \infty \).

**Solution(a):** Since \( u(x) \geq 0 \) for all \( x \), and for \( x \in B_a \) one has \( u(x) > a \), we get

\[
a 1_{B_a}(x) \leq u(x) 1_{B_a}(x) \leq u(x).
\]

for all \( x \in X \) (note that if \( x \notin B_a \), then the above inequalities reduce to \( 0 \leq u(x) \)).

**Solution(b):**

\[
\mu(B_a) = \int 1_{B_a} \, d\mu \leq \frac{1}{a} \int u \, d\mu < \infty.
\]

**Solution(c):** Since \( u(x) > 0 \) for all \( x \in X \), then

\[
X = \bigcup_{n=1}^{\infty} \{ x \in X : u(x) > \frac{1}{n} \} = \bigcup_{n=1}^{\infty} B_{\frac{1}{n}}.
\]

Note that \( (B_{\frac{1}{n}}) \) is an increasing sequence and \( \mu(B_{\frac{1}{n}}) < \infty \) (by part (b)). Thus, \( \mu \) is \( \sigma \)-finite.