Measure and Integration: Solutions Quiz 2013-14

1. Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra on $\mathbb{R}$, and λ is Lebesgue measure.

   (a) Show that any monotonically increasing or decreasing function $f : \mathbb{R} \to \mathbb{R}$ is Borel measurable i.e. $\mathcal{B}(\mathbb{R}) \setminus \mathcal{B}(\mathbb{R})$ measurable. (1.5 pts)

   (b) Show that for any $f \in \mathcal{M}^+(\mathbb{R})$, and any $a \in \mathbb{R}$, one has

   $$\int_{\mathbb{R}} f(x-a)\, d\lambda(x) = \int_{\mathbb{R}} f(x)\, d\lambda(x).$$

   (Hint: start with simple functions.) (1.5 pts)

Proof (a): Assume with no loss of generality that f is monotonically increasing. For any $a \in \mathbb{R}$, consider the set $A_a = \{x \in \mathbb{R} : f(x) > a\}$, and let

$$x_0 = \sup\{x \in \mathbb{R} : f(x) \leq a\}.$$

Notice that

$$A_a = f^{-1}((a,\infty)) = \begin{cases} (x_0,\infty) & \text{if } f(x_0) = a \\ [x_0,\infty) & \text{if } f(x_0) \neq a. \end{cases}$$

By Lemma 8.1, f is Borel measurable.

Proof (b): We apply the standard argument. Suppose first that $f = \mathbf{1}_A$, where $A \in \mathcal{B}(\mathbb{R})$. By translation invariance of Lebesgue measure, we have for any $a \in \mathbb{R}$

$$\int \mathbf{1}_A(x)\, d\lambda(x) = \lambda(A) = \lambda(A+a) = \int \mathbf{1}_{A+a}(x)\, d\lambda(x) = \int \mathbf{1}_A(x-a)\, d\lambda(x).$$

Hence the result is true for indicator functions. Suppose now that $f \in \mathcal{E}^+$, and let $f = \sum_{i=0}^n a_i \mathbf{1}_{A_i}$ be a standard representation. Then

$$\int f(x)\, d\lambda(x) = \sum_{i=0}^n a_i \int \mathbf{1}_{A_i}(x)\, d\lambda(x) = \sum_{i=0}^n a_i \int \mathbf{1}_{A_i}(x-a)\, d\lambda(x) = \int f(x-a)\, d\lambda(x).$$

Now let f be any non-negative measurable function. Then, there exists an increasing sequence $(g_n) \in \mathcal{E}^+$ converging (pointwise) to f. By Beppo-Levi, we have

$$\int f(x)\, d\lambda(x) = \lim_{n\to\infty} \int g_n(x)\, d\lambda(x) = \lim_{n\to\infty} \int g_n(x-a)\, d\lambda(x) = \int f(x-a)\, d\lambda(x).$$

2. Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $(X, \mathcal{A}^*, \overline{\mu})$ be its completion (see exercise 4.13, p.29).

   (a) Show that for any $f \in \mathcal{E}^+(\mathcal{A}^*)$, there exists a function $g \in \mathcal{E}^+(\mathcal{A})$ such that $g(x) \le f(x)$ for all $x \in X$, and

   $$\overline{\mu}(\{x \in X : f(x) \neq g(x)\}) = 0.$$

   (1.5 pts)

   (b) Using Theorem 8.8, show that if $u \in \mathcal{M}^{\pm}_{\overline{\mathbb{R}}}(\mathcal{A}^*)$, then there exists $w \in \mathcal{M}^{\pm}_{\overline{\mathbb{R}}}(\mathcal{A})$ such that $w(x) \le u(x)$ for all $x \in X$, and

   $$\overline{\mu}(\{x \in X : w(x) \neq u(x)\}) = 0.$$

   (1.5 pts)

Proof (a): Let $f = \sum_{i=1}^n a_i \mathbf{1}_{A_i^*}$ be a standard representation of f, with $a_i \ge 0$ and $A_i^* \in \mathcal{A}^*$ pairwise disjoint and $\bigcup_{i=1}^n A_i^* = X$. By Exercise 4.13 (i), for each i there exist $A_i, M_i \in \mathcal{A}$ and $N_i \subseteq M_i$ such that $\mu(M_i) = 0$ and $A_i^* = A_i \cup N_i$. Define $g = \sum_{i=1}^n a_i \mathbf{1}_{A_i}$, then $g \in \mathcal{E}^+(\mathcal{A})$, and $g(x) \le f(x)$ for all $x \in X$. Furthermore,

$$\overline{\mu}(\{x \in X : f(x) \neq g(x)\}) \le \sum_{i=1}^n \mu(M_i) = 0.$$

Proof (b): Let $u \in \mathcal{M}^{\pm}_{\overline{\mathbb{R}}}(\mathcal{A}^*)$. By Theorem 8.8, there exists a sequence $(u_n)_n \in \mathcal{E}^+(\mathcal{A}^*)$ such that $u_n \nearrow u$. By part (a), for each n, there exists $w_n \in \mathcal{E}^+(\mathcal{A})$ with $w_n \le u_n$ and $\overline{\mu}(\{x \in X : w_n(x) \neq u_n(x)\}) = 0$. Let $w = \sup_n w_n$, then $w \le u$, and by Corollary 8.9 we have $w \in \mathcal{M}^{\pm}_{\overline{\mathbb{R}}}(\mathcal{A})$. Finally, since

$$\{x \in X : w(x) \neq u(x)\} \subseteq \bigcup_{n=1}^{\infty} \{x \in X : w_n(x) \neq u_n(x)\},$$

we get

$$\overline{\mu}(\{x \in X : w(x) \neq u(x)\}) \le \sum_{n=1}^{\infty} \overline{\mu}(\{x \in X : w_n(x) \neq u_n(x)\}) = 0.$$

3. Let $(X, \mathcal{B}, \mu)$ be a **finite** measure space and $\mathcal{A}$ be a collection of subsets generating $\mathcal{B}$, i.e. $\mathcal{B} = \sigma(\mathcal{A})$, and satisfying the following conditions: (i) $X \in \mathcal{A}$, (ii) if $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$, and (iii) if $A, B \in \mathcal{A}$, then $A \cup B \in \mathcal{A}$. Let

   $$\mathcal{D} = \{A \in \mathcal{B} : \forall \varepsilon > 0, \exists C \in \mathcal{A} \text{ such that } \mu(A \Delta C) < \varepsilon\}.$$

   (a) Show that if $(A_n)_n \subset \mathcal{D}$ and $\varepsilon > 0$, then there exists a sequence $(C_n)_n \subset \mathcal{A}$ such that

   $$\mu\left(\bigcup_{n=1}^{\infty} A_n \Delta \bigcup_{n=1}^{\infty} C_n\right) < \varepsilon/2.$$

   (1 pt)

(b) Use Theorem 4.4 $(iii)'$ to show that there exists an integer $m \geq 1$ such that

$$\mu\left(\bigcup_{n=1}^{\infty} A_n \Delta \bigcup_{n=1}^{m} C_n\right) < \varepsilon.$$

(1 pt)

(c) Show that $\mathcal{D}$ is a σ-algebra. (1 pt)

(d) Show that $\mathcal{B} = \mathcal{D}$. (1 pt)

Proofs (a), (b) and (c): First note that since $X \in \mathcal{A}$, then $X \in \mathcal{D}$. Now let $A \in \mathcal{D}$ and $\varepsilon > 0$. There exists $C \in \mathcal{A}$ such that $\mu(A\Delta C) < \varepsilon$. Since $C^c \in \mathcal{A}$ and $A\Delta C = A^c \Delta C^c$, we have $\mu(A^c\Delta C^c) < \varepsilon$ and hence $A^c \in \mathcal{D}$. Finally, suppose $(A_n)_n \subset \mathcal{D}$ and $\varepsilon > 0$. For each n, there exists $C_n \in \mathcal{A}$ such that $\mu(A_n\Delta C_n) < \varepsilon/2^{n+1}$. It is easy to check that

$$\bigcup_{n=1}^{\infty} A_n \Delta \bigcup_{n=1}^{\infty} C_n \subseteq \bigcup_{n=1}^{\infty} (A_n \Delta C_n),$$

so that

$$\mu\left(\bigcup_{n=1}^{\infty} A_n \Delta \bigcup_{n=1}^{\infty} C_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n\Delta C_n) < \varepsilon/2.$$

Since $\mathcal{A}$ is closed under finite unions we do not know at this point if $\bigcup_{n=1}^{\infty} C_n$ is an element of $\mathcal{A}$. To solve this problem, we proceed as follows. First note that $\bigcap_{n=1}^{m} C_n^c \searrow \bigcap_{n=1}^{\infty} C_n^c$, hence by Theorem 4.4 $(iii)'$

$$\mu\left(\bigcup_{n=1}^{\infty} A_n \cap \bigcap_{n=1}^{\infty} C_n^c\right) = \lim_{m\to\infty} \mu\left(\bigcup_{n=1}^{\infty} A_n \cap \bigcap_{n=1}^{m} C_n^c\right),$$

and therefore,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n \Delta \bigcup_{n=1}^{\infty} C_n\right) = \lim_{m\to\infty} \mu\left((\bigcup_{n=1}^{\infty} A_n \cap \bigcap_{n=1}^{m} C_n^c) \cup (\bigcap_{n=1}^{\infty} A_n^c \cap \bigcup_{n=1}^{\infty} C_n)\right).$$

Hence there exists m sufficiently large so that

$$\mu\left((\bigcup_{n=1}^{\infty} A_n \cap \bigcap_{n=1}^{m} C_n^c) \cup (\bigcap_{n=1}^{\infty} A_n^c \cap \bigcup_{n=1}^{\infty} C_n)\right) < \mu\left(\bigcup_{n=1}^{\infty} A_n \Delta \bigcup_{n=1}^{\infty} C_n\right) + \varepsilon/2.$$

Since $\bigcap_{n=1}^{\infty} A_n^c \cap \bigcup_{n=1}^{m} C_n \subseteq \bigcap_{n=1}^{\infty} A_n^c \cap \bigcup_{n=1}^{\infty} C_n$, we get

$$\mu\left((\bigcup_{n=1}^{\infty} A_n \cap \bigcap_{n=1}^{m} C_n^c) \cup (\bigcap_{n=1}^{\infty} A_n^c \cap \bigcup_{n=1}^{m} C_n)\right) < \mu\left(\bigcup_{n=1}^{\infty} A_n \Delta \bigcup_{n=1}^{\infty} C_n\right) + \varepsilon/2.$$

Thus,

$$\mu\left((\bigcup_{n=1}^{\infty} A_n \Delta \bigcup_{n=1}^{m} C_n)\right) < \varepsilon,$$

and $\bigcup_{n=1}^{m} C_n \in \mathcal{A}$ since $\mathcal{A}$ is closed under finite unions. This shows that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{D}$. Thus, $\mathcal{D}$ is a σ-algebra.

Proof (d): By definition of $\mathcal{D}$ we have $\mathcal{D} \subseteq \mathcal{B}$. Since $\mathcal{A} \subseteq \mathcal{D}$, and $\mathcal{B}$ is the smallest σ-algebra containing $\mathcal{A}$ we have $\mathcal{B} \subseteq \mathcal{D}$. Therefore, $\mathcal{B} = \mathcal{D}$.