Measure and Integration: Final 2013-14

(1) Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra, and λ Lebesgue measure. Determine the value of

$$\lim_{n\to\infty} \int_{(0,n)} (1+\frac{x}{n})^{-n}(1-\sin\frac{x}{n})\, d\lambda(x).$$

(2 pts)

(2) Let $(X, \mathcal{F}, \mu)$ be a **finite** measure space. Assume $f \in \mathcal{L}^2(\mu)$ satisfies $0 < ||f||_2 < \infty$, and let $A = \{x \in X : f(x) \neq 0\}$. Show that

$$\mu(A) \geq \frac{(\int f\, d\mu)^2}{\int f^2\, d\mu}.$$

(1.5 pts)

(3) Let $E = \{(x, y) : y < x < 1, \, , 0 < y < 1\}$. We consider on E the restriction of the product Borel σ-algebra, and the restriction of the product Lebesgue measure $\lambda \times \lambda$. Let $f : E \to \mathbb{R}$ be given by $f(x, y) = x^{-3/2} \cos(\frac{\pi y}{2x})$.

(a) Show that f is $\lambda \times \lambda$ integrable on E. (0.5 pt)

(b) Define $F : (0, 1) \to \mathbb{R}$ by $F(y) = \int_{(y,1)} x^{-3/2} \cos(\frac{\pi y}{2x})\, d\lambda(x)$. Determine the value of

$$\int F(y)\, d\lambda(y).$$

(2 pts)

(4) Let $1 \leq p < \infty$, and suppose $(X, \mathcal{A}, \mu)$ is a measure space. Let $(f_n)_n \in \mathcal{L}^p(\mu)$ be a sequence converging to f in $\mathcal{L}^p$ i.e. $\lim_{n\to\infty} ||f_n - f||_p = 0$.

(a) Show that

$$\int |f|^p\, d\mu \leq \liminf_{n\to\infty} \int |f_n|^p\, d\mu.$$

(1 pt)

(b) Show that $\lim_{n\to\infty} n^p \mu(\{|f| > n\}) = 0$. (1 pt)

(5) Let $(X, \mathcal{A}, \mu)$ be a finite measure space and $f_n, f \in \mathcal{M}(\mathcal{A})$, $n \geq 1$. Show that f_n converges to f in μ measure **if and only if** $\lim_{n\to\infty} \int \frac{|f_n - f|}{1 + |f_n - f|}\, d\mu = 0$. (2 pts)