(1) Consider the measure space \((\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)\), where \(\mathcal{B}(\mathbb{R})\) is the Borel \(\sigma\)-algebra, and \(\lambda\) Lebesgue measure. Determine the value of
\[
\lim_{n \to \infty} \int_{(0,n)} (1 + \frac{x}{n})^{-n}(1 - \sin \frac{x}{n}) \, d\lambda(x).
\]

(2 pts)

(2) Let \((X, \mathcal{F}, \mu)\) be a finite measure space. Assume \(f \in L^2(\mu)\) satisfies \(0 < ||f||_2 < \infty\), and let \(A = \{x \in X : f(x) \neq 0\}\). Show that
\[
\mu(A) \geq \frac{(\int |f| \, d\mu)^2}{\int f^2 \, d\mu}.
\]
(1.5 pts)

(3) Let \(E = \{(x, y) : y < x < 1, 0 < y < 1\}\). We consider on \(E\) the restriction of the product Borel \(\sigma\)-algebra, and the restriction of the product Lebesgue measure \(\lambda \times \lambda\). Let \(f : E \to \mathbb{R}\) be given by \(f(x, y) = x^{-3/2} \cos(\frac{\pi y}{2x})\).
(a) Show that \(f\) is \(\lambda \times \lambda\) integrable on \(E\). (0.5 pt)
(b) Define \(F : (0, 1) \to \mathbb{R}\) by \(F(y) = \int_{(y,1)} x^{-3/2} \cos(\frac{\pi y}{2x}) \, d\lambda(x)\). Determine the value of
\[
\int F(y) \, d\lambda(y).
\]
(2 pts)

(4) Let \(1 \leq p < \infty\), and suppose \((X, \mathcal{A}, \mu)\) is a measure space. Let \((f_n) \in L^p(\mu)\) be a sequence converging to \(f\) in \(L^p\) i.e. \(\lim_{n \to \infty} ||f_n - f||_p = 0\).
(a) Show that
\[
\int |f|^p \, d\mu \leq \liminf_{n \to \infty} \int |f_n|^p \, d\mu.
\]
(1 pt)
(b) Show that \(\lim_{n \to \infty} n^p \mu(|f| > n) = 0\). (1 pt)

(5) Let \((X, \mathcal{A}, \mu)\) be a finite measure space and \(f_n, f \in \mathcal{M}(\mathcal{A})\), \(n \geq 1\). Show that \(f_n\) converges to \(f\) in \(\mu\) measure if and only if \(\lim_{n \to \infty} \int \frac{|f_n - f|}{1 + |f_n - f|} \, d\mu = 0\). (2 pts)