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Measure and Integration: Quiz 2012-13

1. Consider the measure space $([0,1), \mathcal{B}([0,1)), \lambda)$, where $\mathcal{B}([0,1))$ is the Borel σ-algebra restricted to $[0,1)$ and λ is the restriction of Lebesgue measure on $[0,1)$. Define the transformation $T : [0,1) \to [0,1)$ given by

$$T(x) = \begin{cases} 3x & 0 \le x < 1/3, \\ 3x-1, & 1/3 \le x < 2/3 \\ 3x-2, & 2/3 \le x < 1. \end{cases}$$

   (a) Show that T is $\mathcal{B}([0,1))/\mathcal{B}([0,1))$ measurable.
   (b) Determine the image measure $T(\lambda) = \lambda \circ T^{-1}$.
   (c) Let $\mathcal{C} = \{A \in \mathcal{B}([0,1)) : T^{-1}A = A\}$. Show that $\mathcal{C}$ is a σ-algebra.

2. Let $\mathcal{B}(\mathbb{R}^n)$ be the Borel σ-algebra over $\mathbb{R}^n$, and let $\overline{\mathcal{B}}(\mathbb{R}^n)$ be the completion of $\mathcal{B}(\mathbb{R}^n)$ (In the notation of exercise 4.13, p.29, if $\mathcal{A} = \mathcal{B}(\mathbb{R}^n)$, then $\mathcal{A}^* = \overline{\mathcal{B}}(\mathbb{R}^n)$). The σ-algebra $\overline{\mathcal{B}}(\mathbb{R}^n)$ is called the Lebesgue σ-algebra over $\mathbb{R}^n$. Let $n = 1$ and suppose $M \subset \mathbb{R}$ is a **non**-Lebesgue measurable set (i.e. $M \notin \overline{\mathcal{B}}(\mathbb{R})$). Define $A = \{(x,x) \in \mathbb{R}^2 : x \in M\}$, and let $g : \mathbb{R} \to \mathbb{R}^2$ be given by $g(x) = (x,x)$.

   (a) Show that $A \in \overline{\mathcal{B}}(\mathbb{R}^2)$ i.e. A is Lebesgue measurable.
   (b) Show that g is a Borel measurable function, i.e. $g^{-1}(B) \in \mathcal{B}(\mathbb{R})$ for each $B \in \mathcal{B}(\mathbb{R}^2)$.
   (c) Show that $A \notin \mathcal{B}(\mathbb{R}^2)$, i.e. A is not Borel measurable.

3. Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra over $\mathbb{R}$, and λ is Lebesgue measure. Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$f_n(x) = \sum_{k=0}^{2^n-1} \frac{k}{2^n} \cdot 1_{[k/2^n,(k+1)/2^n)},\ n \ge 1.$$

   (a) Show that f_n is measurable, and $f_n(x) \le f_{n+1}(x)$ for all $x \in X$.
   (b) Let $f(x) = x\mathbf{1}_{[0,1)}(x)$. Show that f is $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$ measurable.
   (c) Prove that $f(x) = \lim_{n\to\infty} f_n(x) = \sup_{n\ge1} f_n(x)$ for all $x \in \mathbb{R}$.
   (d) Show that $\int f d\lambda = \frac{1}{2}$.

4. Let $(X, \mathcal{A}, \mu)$ be a measure space, and $u \in \mathcal{M}^{\pm}_{\mathbb{R}}(\mathcal{A})$ satisfying $\int u\, d\mu < \infty$. For $a > 0$ (a real number) set $B_a = \{x \in X : u(x) > a\}$.

(a) Show that for any $a > 0$ one has

$$a\mathbf{1}_{B_a}(x) \leq u(x) \text{ for all } x \in X.$$

(b) Prove that $\mu(B_a) < \infty$ for all $a > 0$.

(c) Assume that $u(x) > 0$ for all $x \in X$, i.e. u is strictly positive. Show that μ is σ-finite, i.e. there exists an exhausting sequence $A_n \nearrow X$ with $\mu(A_n) < \infty$.