1. Consider the measure space \((\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)\), where \(\mathcal{B}(\mathbb{R})\) is the Borel \(\sigma\)-algebra on \(\mathbb{R}\), and \(\lambda\) is Lebesgue measure.

(a) Show that any monotonically increasing or decreasing function \(f : \mathbb{R} \to \mathbb{R}\) is Borel measurable i.e. \(\mathcal{B}(\mathbb{R}) \setminus \mathcal{B}(\mathbb{R})\) measurable. (1.5 pts)

(b) Show that for any \(f \in \mathcal{M}^+(\mathbb{R})\), and any \(a \in \mathbb{R}\), one has

\[
\int_{\mathbb{R}} f(x - a) \, d\lambda(x) = \int_{\mathbb{R}} f(x) \, d\lambda(x).
\]

(Hint: start with simple functions.) (1.5 pts)

2. Let \((X, \mathcal{A}, \mu)\) be a measure space, and let \((X, \mathcal{A}^*, \overline{\mu})\) be its completion (see exercise 4.13, p.29).

(a) Show that for any \(f \in \mathcal{E}^+(\mathcal{A}^*)\), there exists a function \(g \in \mathcal{E}^+(\mathcal{A})\) such that \(g(x) \leq f(x)\) for all \(x \in X\), and

\[
\overline{\mu}(\{x \in X : f(x) \neq g(x)\}) = 0.
\]

(1.5 pts)

(b) Using Theorem 8.8, show that if \(u \in \mathcal{M}^+_{\mathbb{R}}(\mathcal{A}^*)\), then there exists \(w \in \mathcal{M}^+_{\mathbb{R}}(\mathcal{A})\) such that \(w(x) \leq u(x)\) for all \(x \in X\), and

\[
\overline{\mu}(\{x \in X : w(x) \neq u(x)\}) = 0.
\]

(1.5 pts)

3. Let \((X, \mathcal{B}, \mu)\) be a finite measure space and \(\mathcal{A}\) be a collection of subsets generating \(\mathcal{B}\), i.e. \(\mathcal{B} = \sigma(\mathcal{A})\), and satisfying the following conditions: (i) \(X \in \mathcal{A}\), (ii) if \(A \in \mathcal{A}\), then \(A^c \in \mathcal{A}\), and (iii) if \(A, B \in \mathcal{A}\), then \(A \cup B \in \mathcal{A}\). Let

\[
\mathcal{D} = \{A \in \mathcal{B} : \forall \varepsilon > 0, \exists C \in \mathcal{A} \text{ such that } \mu(A \Delta C) < \varepsilon\}.
\]

(a) Show that if \((A_n)_n \subset \mathcal{D}\) and \(\varepsilon > 0\), then there exists a sequence \((C_n)_n \subset \mathcal{A}\) such that

\[
\mu \left( \bigcup_{n=1}^{\infty} A_n \Delta \bigcup_{n=1}^{\infty} C_n \right) < \varepsilon/2.
\]

(1 pt)

(b) Use Theorem 4.4 (iii)' to show that there exists an integer \(m \geq 1\) such that

\[
\mu \left( \bigcup_{n=1}^{\infty} A_n \Delta \bigcup_{n=1}^{m} C_n \right) < \varepsilon.
\]

(1 pt)

(c) Show that \(\mathcal{D}\) is a \(\sigma\)-algebra. (1 pt)

(d) Show that \(\mathcal{B} = \mathcal{D}\). (1 pt)