JUSTIFY YOUR ANSWERS
Allowed: calculator, material handed out in class and handwritten notes (your handwriting). NO BOOK IS ALLOWED

NOTE:
• The test consists of seven exercises for a total of 11 credits.
• The score is computed by adding all the credits up to a maximum of 10

Exercise 1. Consider events $A, B$ of a probability space. Prove the following:
(a) (0.5 pts.) If $\mathbb{P}(B) = 0$ then $\mathbb{P}(A \cup B) = \mathbb{P}(A)$.
(b) (0.5 pts.) If $\mathbb{P}(B) = 1$ then $\mathbb{P}(A \cap B) = \mathbb{P}(A)$.

Exercise 2. Let $X_1, X_2, \ldots$, be independent identically distributed normal variables with mean $\mu$ and standard deviations $\sigma^2$. Let $S_n = \sum_{i=1}^{n} X_i$.
(a) (0.6 pts.) Find the moment-generating function for $S_n$ and identify the law of $S_n$.
(b) (0.8 pts.) Let $N$ be a Poisson$(\lambda)$ random variable. Compute the moment-generating function of $S_N$.

Exercise 3. Consider a random walk in $0, 1, \ldots, N$ with probability $p$ of moving to the right, $q$ of moving to the left and partial reflections at the endpoints. That is, a Markov process with transition probabilities
$$
\begin{align*}
P_{0,0} &= 1 - p \\
P_{0,1} &= p \\
P_{i,i+1} &= p & i = 1, \ldots, N - 1 \\
P_{i,i} &= 1 - p - q & i = 1, \ldots, N - 1 \\
P_{i,i-1} &= q & i = 1, \ldots, N - 1 \\
P_{N,N-1} &= q \\
P_{N,N} &= 1 - q
\end{align*}
$$
(a) (0.8 pts.) If initially the walker is at position 1, which is the probability that after 2 time units the walker is again in 1.
(b) (0.8 pts.) Find the reversible (hence invariant) probability for this process for all values of $p \in (0, 1)$.

Exercise 4. The ticket office of Leiden Central Station has two clerks whose service time are independent and exponentially distributed with respective rates $\lambda_1$ and $\lambda_2$. When customer C arrives, she finds both clerks busy, one with customer $A$ and the other with customer $B$. He will be serviced by the first clerk that completes service to his current customer. Find:
(a) (0.8 pts.) The mean time needed for $C$ to complete her ticket purchase.
(b) (0.8 pts.) The probability that $C$ is not the last to leave the ticket office.

Exercise 5. Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate $\lambda$. Find
Exercise 6. A piramidal molecule undergoes oscillations in which one of the atoms changes from an Up to a Bottom position, passing through a Centre position. Simulations show that the times the atom stays at each position are exponentially distributed with mean 2 (picoseconds) for the Up and Bottom positions and mean 1 for the Centre position. An atom in the Up or Bottom position always moves towards the Centre after leaving its position, while an atom in the Centre moves towards the bottom twice as often as towards the top.

(a) (0.8 pts.) Model this evolution as a continuous-time Markov chain among the positions \( T \), \( C \) and \( B \). That is, determine the abandoning rates \( \nu_P \), \( \nu_C \) and \( \nu_B \) and the transitions \( P_{ij} \) with \( i,j = T,C,B \).

(b) (0.8 pts.) Determine, in the long run, the fraction of time spent by the atom in each of the three positions.

Exercise 7. Consider a pure death process with three states, that is, a process whose only non-zero rates are the death rates \( \mu_1 \) and \( \mu_2 \).

(a) (0.8 pts.) Write the five non-trivial forward Kolmogorov equations.

(b) (0.6 pts.) Find \( P_{11}(t) \) and \( P_{22}(t) \).