Geometry and Topology – Exam 3

Notes:
1. **Write your name and student number clearly on each page of written solutions you hand in.**
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are allowed to consult text books and class notes.
5. You are not allowed to consult colleagues, calculators, computers etc.
6. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

Questions

**Exercise 1** (1.5 pt). In each list of spaces below, decide which spaces are homotopy equivalent to each other (remember to justify your answer)

a) $K\#T^2$, $\mathbb{R}P^2\#T^2\#S^2$, $K\#\mathbb{R}P^2\#\mathbb{R}P^2$, $\mathbb{R}P^2\#\mathbb{R}P^2\#\mathbb{R}P^2$.

where $K$ denotes the Klein bottle and $T^2$ is the 2-dimensional torus.

b) $S^{2n}$, $\mathbb{R}P^{2n}$, $\mathbb{C}P^n$, for $n > 1$.

**Exercise 2** (1.0 pt). A deformation retraction in the weak sense of a space $X$ to a subspace $A \subset X$ is a homotopy $f_t : X \rightarrow X$ such that $f_0 = \text{Id}$, $f_1(X) \subset A$ and $f_t(A) \subset A$ for all $t \in [0,1]$. Show that if $X$ deformation retracts to $A$ in this weak sense, then the inclusion $A \hookrightarrow X$ is a homotopy equivalence.

**Exercise 3** (1.0 pt). Let $X$ be the quotient space of $S^2$ obtained by identifying the north and south poles to a single point. Put a cell complex structure on $X$ and use this to compute $\pi_1(X)$.

**Exercise 4** (2.0 pt). Let $X$ and $Y$ be path connected and locally path connected and let $\tilde{X}$ and $\tilde{Y}$ be their simply-connected covering spaces, respectively. Show that if $X$ is homotopy equivalent to $Y$ then $\tilde{X}$ is homotopy equivalent to $\tilde{Y}$.

**Exercise 5** (1.0 pt). Show that if $A \subset X$ is a retract of $X$ then the map $H_n(A) \rightarrow H_n(X)$ induced by the inclusion $A \hookrightarrow X$ is injective for all $n$.

**Exercise 6** (1.5 pt).

a) Given a map $f : S^{2n} \rightarrow S^{2n}$, show that there is some point $x \in S^{2n}$ with either $f(x) = x$ or $f(x) = -x$. 
b) Show that every map \( f : \mathbb{R}P^{2n} \to \mathbb{R}P^{2n} \) has a fixed point.

c) Construct a map \( f : \mathbb{R}P^{2n-1} \to \mathbb{R}P^{2n-1} \) without fixed points. (Hint: use linear a transformation \( \mathbb{R}^2 \to \mathbb{R}^2 \) without eigenvectors.

**Exercise 7** (2.0 pt). Show that \( H_i(X \times S^n) = H_i(X) \oplus H_{i-n}(X) \) for all \( i \) and \( n \), where \( H_i = 0 \) for \( i < 0 \) by definition. Hint: show \( H_i(X \times S^n) = H_i(X) \oplus H_i(X \times S^n; X \times \{x_0\}) \) and \( H_i(X \times S^n; X \times \{x_0\}) = H_{i-1}(X \times S^{n-1}; X \times \{x_0\}) \).