Geometry and Topology – Exam 1

Notes:
1. Write your name and student number **clearly** on each page of written solutions you hand in.
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are allowed to consult text books and class notes.
5. You are not allowed to consult colleagues, calculators, computers etc.
6. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

Questions

Exercise 1 (2.0 pt). In each list of spaces below, decide which spaces are homotopy equivalent to each other (remember to justify your answer)

a) $S^1 \times S^1 \setminus \{p\}$, $S^1 \vee S^1$, $K\setminus\{p\}$, $S^2\setminus\{p_1,p_2,p_3\}$
where $K$ denotes the Klein bottle.

b) $\mathbb{R}\setminus\{0\}$, $\mathbb{R}^2\setminus\{0\}$, $\mathbb{R}^3\setminus\{0\}$

Exercise 2 (2.0 pt).

a) Show that a retract of a contractible space is contractible.

b) Let $f : X \to Y$ be continuous. Show that if there are maps $g, h : Y \to X$ such that $f \circ g \simeq \text{Id}_Y$ and $h \circ f \simeq \text{Id}_X$ then $f$ is a homotopy equivalence.

c) Show that if $X = X_1 \cup X_2$ is a CW complex, $X_1$, $X_2$ are subcomplexes and $X_1$, $X_2$ and $X_1 \cap X_2$ are contractible then $X$ is contractible.

Exercise 3 (2.0 pt). Show that for a space $X$, the following three statements are equivalent:

a) Every map $S^1 \to X$ is homotopic to a constant map.

b) Every map $S^1 \to X$ extends to a map $D^2 \to X$.

c) $\pi_1(X, x_0) = \{0\}$ for all $x_0 \in X$.

Exercise 4 (3.0 pt).
a) Show that if a space $X$ is obtained from a path connected space $X_0$ by attaching $n$-cells with $n > 2$ then the inclusion $\iota : X_0 \hookrightarrow X$ induces an isomorphism of fundamental groups: $\iota_* : \pi_1(X_0, x_0) \cong \pi_1(X, x_0)$.

b) Let $X \in \mathbb{R}^n$ be the union of convex open sets $\{X_1, \cdots, X_m\}$ such that $X_i \cup X_j \cup X_k \neq \emptyset$ for all $i, j, k$. Show that $X$ is simply connected.

**Exercise 5** (1.0 pt). Show that if $X$ is path connected and locally path connected space and $\pi_1(X)$ is finite then every map $f : X \to S^1$ is null homotopic.