Geometry and Topology – Exam 2

Notes:

1. **Write your name and student number clearly on each page of written solutions you hand in.**

2. You can give solutions in English or Dutch.

3. You are expected to explain your answers.

4. You are allowed to consult text books and class notes.

5. You are **not** allowed to consult colleagues, calculators, computers etc.

6. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

Questions

Exercise 1 (2.0 pt). In each list of spaces below, decide which spaces are homotopy equivalent to each other (remember to justify your answer)

a) \( S^1 \times S^n \) and \( S^1 \lor S^n \lor S^{n+1} \) for \( n > 1 \),

b) \( \mathbb{R}P^2 \# \mathbb{R}P^2 \# T^2 \), \( \mathbb{K} \# \mathbb{K} \), \( T^2 \# T^2 \).

where \( \mathbb{K} \) denotes the Klein bottle and \( T^2 \) is the 2-dimensional torus.

c) \( S^{2n} \), \( \mathbb{R}P^{2n} \), \( \mathbb{C}P^n \) for \( n > 1 \).

Exercise 2 (2.0 pt). Let \( p : \tilde{X} \to X \) be a simply connected cover of the space \( X \) and let \( A \subset X \) be a path connected and locally path connected subspace. Let \( \tilde{A} \subset \tilde{X} \) be a path component of \( p^{-1}(A) \). Show that \( p : \tilde{A} \to A \) is the covering space corresponding to the kernel of the map \( \pi_1(A) \to \pi_1(X) \).

Exercise 3 (1.0 pt). The suspension of a set \( X \) is the quotient of \( I \times X \) by the equivalence relation \( (0, x) \sim (0, x') \) and \( (1, x) \sim (1, x') \) for all \( x, x' \in X \). Denoting by \( SX \) be the suspension of \( X \), show that \( \tilde{H}_n(X) = \tilde{H}_{n+1}(SX) \).

Exercise 4 (2.0 pt). Let \( \mathcal{U} = \{U_1, \cdots, U_k\} \) be an open cover of a space \( X \) with the following properties

- All the intersections of the form \( U_{i_0} \cap \cdots \cap U_{i_l} \) are either contractible or empty (in particular, each \( U_i \) is contractible);
- There is an \( n > 0 \) for which \( U_{i_0} \cap \cdots \cap U_{i_n} = \emptyset \) for all possible choices of distinct indices.
Show that $H_i(X) = \{0\}$ for all $i \geq n$.

Note: you are not allowed to use Čech cohomology to prove this claim.

**Exercise 5 (3.0 pt).** For each statement below, prove or give a counter example.

a) If $f : X \to Y$ is a homotopy equivalence and $x \in X$ then $X\setminus\{x\}$ and $Y\setminus\{f(x)\}$ are homotopy equivalent.

b) If $\pi_1(X)$ is finite and $X$ is compact, then every path connected covering space of $X$ is compact.

c) Let $X$ be path connected and locally path connected and $\tilde{X}$ be a path connected cover of $X$ with covering map $p : \tilde{X} \to X$. Then $p_* : H_k(\tilde{X}) \to H_k(X)$ is an injection for all $k$. 