Measure and Integration: Solutions Final 2014-15

1. Consider a measure space \((X, \mathcal{A}, \mu)\), and let \((f_n)\) be a sequence in \(L^2(\mu)\) which is bounded in the \(L^2\) norm, i.e. there exists a constant \(C > 0\) such that \(\|f_n\|_2 < C\) for all \(n \geq 1\).
   
   (a) Prove that \(\sum_{n=1}^{\infty} (f_n/n)^2 \in L_1^1(\mu)\). (1 pt.)
   
   (b) Prove that \(\lim_{n \to \infty} f_n/n = 0\) \(\mu\) a.e. (1 pt.)

**Proof (a):** First observe that
\[
\sum_{n=1}^{\infty} \|f_n/n\|^2_2 = \sum_{n=1}^{\infty} \frac{\|f_n\|^2_2}{n^2} \leq \sum_{n=1}^{\infty} \frac{C^2}{n^2} < \infty.
\]
Now, by Beppo-Levi and the above, we have
\[
\int \sum_{n=1}^{\infty} (f_n/n)^2 \, d\mu = \sum_{n=1}^{\infty} \int (f_n/n)^2 \, d\mu = \sum_{n=1}^{\infty} \frac{\|f_n\|^2_2}{n^2} < \infty.
\]
Hence, \(\sum_{n=1}^{\infty} (f_n/n)^2 \in L_1^1(\mu)\).

**Proof (b):** Since \(\sum_{n=1}^{\infty} (f_n/n)^2 \in L_1^1(\mu)\), then \(\sum_{n=1}^{\infty} (f_n/n)^2 < \infty\) \(\mu\) a.e. and as a result \(\lim_{n \to \infty} (f_n/n) = 0\) \(\mu\) a.e.

2. Let \((X, \mathcal{A}, \mu)\) be a finite measure space. Suppose that the real valued functions \(f_n, g_n, f, g \in \mathcal{M}(\mathcal{A})\) \((n \geq 1)\) satisfy the following:
   
   (i) \(f_n \overset{\mu}{\to} f\),
   
   (ii) \(g_n \overset{\mu}{\to} g\),
   
   (iii) \(|f_n| \leq C\) for all \(n\), where \(C > 0\).

   Prove that \(f_n g_n \overset{\mu}{\to} f g\). (2 pts)

**Proof:** Let \(\epsilon > 0\) and \(\delta > 0\), since \(\mu\) is a finite measure, it is enough to show that there exists \(N \geq 1\) such that
\[
\mu(\{x \in X : |f_n g_n - f g| > \epsilon\}) < \delta, \quad \text{for all } n \geq N.
\]

First note that
\[
|f_n g_n - f g| \leq |f_n| g_n - g| + |g| |f_n - f|,
\]
thus,
\[
\mu(\{x \in X : |f_n g_n - f g| > \epsilon\}) \leq \mu(\{x \in X : |f_n| g_n - g| > \epsilon/2\}) + \mu(\{x \in X : |g| |f_n - f| > \epsilon/2\}).
\]

Let \(E_n = \{x \in X : |g| > n\}\), then \(E_1 \supseteq E_2 \supseteq \cdots\), and since \(g\) is real valued we have \(\bigcap_{n=1}^{\infty} E_n = \emptyset\). By finiteness of \(\mu\), we have
\[
\lim_{n \to \infty} \mu(E_n) = \mu(\bigcap_{n=1}^{\infty} E_n) = 0.
\]

Choose \(m\) large enough so that \(\mu(E_m) < \delta/3\). By properties (i) and (ii), there exists \(N \geq 1\) so that for \(n \geq N\),
\[
\mu(\{x \in X : |f_n - f| > \epsilon/2m\}) < \delta/3, \quad \text{and} \quad \mu(\{x \in X : |g_n - g| > \epsilon/2C\}) < \delta/3.
\]

Then for all \(n \geq N\),
\[
\mu(\{x \in X : |f_n| g_n - g| > \epsilon/2\}) \leq \mu(\{x \in X : |g_n - g| > \epsilon/2C\}) < \delta/3,
\]
and
\[
\mu(\{x \in X : |g| |f_n - f| > \epsilon/2\}) \leq \mu(E_m) + \mu(\{x \in E_m^c : |f_n - f| > \epsilon/2m\}) < 2\delta/3.
\]

Therefore, \(\mu(\{x \in X : |f_n g_n - f g| > \epsilon\}) < \delta\) for all \(n \geq N\), and hence \(f_n g_n \overset{\mu}{\to} f g\).
Let \((X, \mathcal{A})\) be a measurable space and let \(\mu, \nu\) be finite measures on \(\mathcal{A}\).

(a) Show that there exists a function \(f \in L^1(\mu) \cap L^1(\nu)\) such that for every \(A \in \mathcal{A}\), we have
\[
\int_A (1 - f) \, d\mu = \int_A f \, d\nu.
\]
(1 pt)

(b) Show that the function \(f\) of part (a) satisfies \(0 \leq f \leq 1\) a.e. (1 pt)

**Proof (a):** First note that \(\mu + \nu\) is a measure (Exercise 4.6(ii)), and that \(\mu \ll \mu + \nu\). By using a standard argument (first checking indicator functions, then simple functions, then positive functions, then general integrable functions) one sees that for any \(g \in L^1(\mu + \nu)\) one has \(g \in L^1(\mu) \cap L^1(\nu)\), and
\[
\int g \, d(\mu + \nu) = \int g \, d\mu + \int g \, d\nu.
\]

Now the condition \(\int_A (1 - f) \, d\mu = \int_A f \, d\nu\) is equivalent to \(\mu(A) = \int_A f \, d(\mu + \nu)\). Since \(\mu \ll \mu + \nu\), then by Radon-Nikodym Theorem there exists \(f \in L^1(\mu + \nu)\) such that \(\mu(A) = \int_A f \, d(\mu + \nu)\). Thus, \(f \in L^1(\mu) \cap L^1(\nu)\) and \(\int_A (1 - f) \, d\mu = \int_A f \, d\nu\) for all \(A \in \mathcal{A}\).

**Proof (b):** Define \(\rho\) on \(\mathcal{A}\) by \(\rho(A) = \int_A f \, d\mu\) \((A \in \mathcal{A})\). Since \(f \in L^1(\nu)\), then \(\rho\) is a finite measure and \(\rho \ll \nu\). By part (a), we have \(\rho(A) = \int_A (1 - f) \, d\mu\) \((A \in \mathcal{A}\) and \((1 - f) \in L^1(\mu)\). By Theorem 10.9(ii), we see that if \(\mu(A) = 0\), then \(\rho(A) = 0\), hence \(\rho \ll \mu\). By the Theorem of Radon Nikodym, there exists a unique a.e. function \(g \in L^1(\mu)\) such that \(\rho(A) = \int_A g \, d\mu\) for all \(A \in \mathcal{A}\). This gives that
\[
\int_A g \, d\mu = \int_A (1 - f) \, d\mu, \text{ for all } A \in \mathcal{A}.
\]

By Corollary 10.14(i), we have \(g = 1 - f\) a.e. Since \(g, f \geq 0\), we get \(0 \leq f \leq 1\) a.e.

Let \(0 < a < b\). Prove with the help of Tonelli’s theorem (applied to the function \(f(x, t) = e^{-xt}\)) that \(\int_{[0, \infty)} (e^{-at} - e^{-bt}) \frac{1}{t} \, d\lambda(t) = \log(b/a)\), where \(\lambda\) denotes Lebesgue measure. (2 pts)

**Proof** Let \(f : [a, b] \times [0, \infty)\) be given by \(f(x, t) = e^{-xt}\). Then \(f\) is continuous (hence measurable) and \(f > 0\). By Tonelli’s theorem
\[
\int_{[0, \infty)} \int_{[a, b]} e^{-xt} \, d\lambda(x) \, d\lambda(t) = \int_{[a, b]} \int_{[0, \infty)} e^{-xt} \, d\lambda(t) \, d\lambda(x).
\]
For each fixed \(x \in [a, b]\), the function \(t \to e^{-xt}\) is positive measurable and the improper Riemann integrable on \([0, \infty)\) exists, so that
\[
\int_{[0, \infty)} e^{-xt} \, d\lambda(t) = \int_0^\infty e^{-xt} \, dt = \frac{1}{x}.
\]

Furthermore, the function \(x \to \frac{1}{x}\) is measurable and Riemann integrable on \([a, b]\), thus
\[
\int_{[a, b]} \int_{[0, \infty)} e^{-xt} \, d\lambda(t) \, d\lambda(x) = \int_{[a, b]} \frac{1}{x} \, d\lambda(x) = \int_a^b \frac{1}{x} \, dx = \log(b/a).
\]

On the other hand,
\[
\int_{[0, \infty)} \int_{[a, b]} e^{-xt} \, d\lambda(x) \, d\lambda(t) = \int_{[0, \infty)} \int_a^b e^{-xt} \, dx \, d\lambda(t) = \int_{[0, \infty)} (e^{-at} - e^{-bt}) \frac{1}{t} \, d\lambda(t).
\]

Therefore, \(\int_{[0, \infty)} (e^{-at} - e^{-bt}) \frac{1}{t} \, d\lambda(t) = \log(b/a)\).

Let \((X, \mathcal{A}, \mu)\) be a finite measure space, and \(f \in M(\mathcal{A})\) satisfies \(f^n \in L^1(\mu)\) for all \(n \geq 1\).

(a) Show that if \(\lim_{n \to \infty} \int f^n \, d\mu\) exists and is finite, then \(|f(x)| \leq 1\) a.e. (1 pt)

(b) Show that \(\int f^n \, d\mu = c\) is a constant for all \(n \geq 1\) if and only if \(f = 1_A \mu\) a.e. for some measurable set \(A \in \mathcal{A}\). (1 pt)
Proof (a) Let \( E = \{ x \in X : |f(x)| > 1 \} \) and assume for the sake of getting a contradiction that \( \mu(E) > 0 \). For \( k \geq 1 \), let \( E_k = \{ x \in X : |f(x)| > 1 + 1/k \} \). Then \( E_k \) is an increasing sequence of measurable set with \( E = \bigcup_{k=1}^{\infty} E_k \). Since \( \mu(E) > 0 \), there exists \( k \geq 1 \) sufficiently large such that \( \mu(E_k) > 0 \). Note that for any \( n \geq 1 \),

\[
f^{2n} = f^{2n}1_{E_k} + f^{2n}1_{E_k^c} \geq f^{2n}1_{E_k} \geq (1 + 1/k)^{2n}1_{E_k}.
\]

Thus, for all \( n \geq 1 \)

\[
\int f^{2n} \, d\mu \geq (1 + 1/k)^{2n} \mu(E_k).
\]

This implies that

\[
\lim_{n \to \infty} \int f^{2n} \, d\mu \geq \lim_{n \to \infty} (1 + 1/k)^{2n} \mu(E_k) = \infty,
\]

contradicting the fact that \( \lim_{n \to \infty} \int f^n \, d\mu < \infty \). Thus \( \mu(E) = 0 \) and \( |f(x)| \leq 1 \) \( \mu \)-a.e.

Proof (b) If \( f = 1_A \) for some measurable set \( A \in \mathcal{A} \), then \( f^n = 1_A \) for all \( n \geq 1 \) and hence

\[
\int f^n \, d\mu = \mu(A) \text{ for all } n \geq 1.
\]

Conversely, assume \( \int f^n \, d\mu = c \) for all \( n \geq 1 \). Since \( \lim_{n \to \infty} \int f^n \, d\mu = c \) exists and is finite, then by part (a), we have that \( |f(x)| \leq 1 \) \( \mu \)-a.e. Let \( A = \{ x \in X : f(x) = 1 \} \), \( B = \{ x \in X : f(x) = -1 \} \), and \( C = \{ x \in X : |f(x)| < 1 \} \). Since \( f \in \mathcal{M}(A) \), then \( A, B, C \in \mathcal{A} \), and \( f = 1_A f + 1_B f + 1_C f \) and for any \( n \geq 1 \),

\[
c = \int f^n \, d\mu = \mu(A) + (-1)^n \mu(B) + \int_C f^n \, d\mu,
\]

as well as

\[
c = \lim_{n \to \infty} \int f^n \, d\mu = \lim_{n \to \infty} \left( \mu(A) + (-1)^n \mu(B) + \int_C f^n \, d\mu \right).
\]

Note that \( \lim_{n \to \infty} 1_C f^n(x) = 0 \) for all \( x \in X \), and \( |1_C f^n(x)| \leq 1 \). Since \( \mu(X) < \infty \), then \( 1 \in L^1(\mu) \), then by Lebesgue Dominated Convergence Theorem,

\[
\lim_{n \to \infty} \int_C f^n \, d\mu = \lim_{n \to \infty} \int 1_C f^n \, d\mu = \int \lim_{n \to \infty} 1_C f^n \, d\mu = 0.
\]

As a result we have

\[
c = \lim_{n \to \infty} \left( \mu(A) + (-1)^n \mu(B) \right).
\]

If we take the limit along even \( n \), we get \( c = \mu(A) + \mu(B) \), and if we take the limit along odd \( n \), we get \( c = \mu(A) - \mu(B) \). This implies that \( \mu(B) = 0 \), and hence \( 1_B f = 0 \) \( \mu \)-a.e. Therefore, \( c = \mu(A) = \mu(A) + \int_C f^n \, d\mu = 0 \) for all \( n \geq 1 \), and hence \( \int_C f^n \, d\mu = 0 \) for all \( n \geq 1 \). In particular, \( \int_C f^2 \, d\mu = \int 1_C f^2 \, d\mu = 0 \). Since \( 1_C f^2 \geq 0 \), this implies that \( 1_C f^2 = 0 \) \( \mu \)-a.e. and hence \( 1_C f = 0 \) \( \mu \)-a.e. Thus, \( f = 1_A f + 1_B f + 1_C f = 1_A f = 1_A \) \( \mu \)-a.e.

We give also a second short proof: Note that

\[
\int f^2 (1 - f)^2 \, d\mu = \int f^2 \, d\mu - 2 \int f^3 \, d\mu + \int f^4 \, d\mu = c - 2c + c = 0.
\]

Since \( f^2 (1 - f)^2 \geq 0 \), this implies that \( f^2 (1 - f)^2 = 0 \) \( \mu \)-a.e. implying that \( f \) is 0 or 1 \( \mu \)-a.e. equivalently \( f \) is \( \mu \)-a.e.equally the indicator function \( 1_A \) with \( A = \{ x \in X : f(x) = 1 \} \).