Measure and Integration: Final 2014-15

(1) Consider a measure space \((X, \mathcal{A}, \mu)\), and let \((f_n)\) be a sequence in \(L^2(\mu)\) which is bounded in the \(L^2\) norm, i.e. there exists a constant \(C > 0\) such that \(|f_n|_2 < C\) for all \(n \geq 1\).

(a) Prove that \(\sum_{n=1}^{\infty} (\frac{f_n}{n})^2 \in L_1(\mu)\). (1 pt.)

(b) Prove that \(\lim_{n \to \infty} \frac{f_n}{n} = 0\) \(\mu\) a.e. (1 pt.)

(2) Let \((X, \mathcal{A}, \mu)\) be a finite measure space. Suppose that the real valued functions \(f_n, g_n, f, g \in \mathcal{M}(\mathcal{A})\) \((n \geq 1)\) satisfy the following:

(i) \(f_n \xrightarrow{\mu} f\),
(ii) \(g_n \xrightarrow{\mu} g\),
(iii) \(|f_n| \leq C\) for all \(n\), where \(C > 0\).

Prove that \(f_n g_n \xrightarrow{\mu} fg\). (2 pts)

(3) Let \((X, \mathcal{A})\) be a measurable space and let \(\mu, \nu\) be finite measures on \(\mathcal{A}\).

(a) Show that there exists a function \(f \in L^1_+(\mu) \cap L^1_+(\nu)\) such that for every \(A \in \mathcal{A}\), we have

\[
\int_A (1 - f) \, d\mu = \int_A f \, d\nu.
\]

(1 pt)

(b) Show that the function \(f\) of part (a) satisfies \(0 \leq f \leq 1\) \(\mu\) a.e. (1 pt)

(4) Let \(0 < a < b\). Prove with the help of Tonelli’s theorem (applied to the function \(f(x,t) = e^{-xt}\)) that

\[
\int_{[0,\infty)} e^{-at} - e^{-bt} \frac{1}{t} \, d\lambda(t) = \log(b/a),
\]

where \(\lambda\) denotes Lebesgue measure. (2 pts)

(5) Let \((X, \mathcal{A}, \mu)\) be a finite measure space, and \(f \in \mathcal{M}(\mathcal{A})\) satisfies \(f^n \in L^1(\mu)\) for all \(n \geq 1\).

(a) Show that if \(\lim_{n \to \infty} \int f^n \, d\mu\) exists and is finite, then \(|f(x)| \leq 1\) \(\mu\) a.e. (1 pt)

(b) Show that \(\int f^n \, d\mu = c\) is a constant for all \(n \geq 1\) if and only if \(f = 1_A\) \(\mu\) a.e. for some measurable set \(A \in \mathcal{A}\). (1 pt)