Measure and Integration: Quiz 2014-15

1. Consider the measure space \((\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)\), where \(\mathcal{B}(\mathbb{R})\) is the Borel \(\sigma\)-algebra over \(\mathbb{R}\), and \(\lambda\) is Lebesgue measure. Let \(f_n : \mathbb{R} \to \mathbb{R}\) be defined by

\[
f_n(x) = \sum_{k=0}^{2^n-1} 3k + 2^n \cdot \mathbf{1}_{[k/2^n,(k+1)/2^n]}(x), \quad n \geq 1.
\]

(a) Show that \(f_n\) is measurable, and \(f_n(x) \leq f_{n+1}(x)\) for all \(x \in \mathbb{R}\). (1 pt)

(b) Show that \(\int \sup_{n\geq1} f_n \, d\lambda = \frac{5}{2}\). (2 pts)

2. Let \(X\) be a set, and \(\mathcal{C} \subseteq \mathcal{P}(X)\). Consider \(\sigma(\mathcal{C})\), the smallest \(\sigma\)-algebra over \(X\) containing \(\mathcal{C}\), and let \(D\) be the collection of sets \(A \in \sigma(\mathcal{C})\) with the property that there exists a countable collection \(\mathcal{C}_0 \subseteq \mathcal{C}\) (depending on \(A\)) such that \(A \in \sigma(\mathcal{C}_0)\).

(a) Show that \(D\) is a \(\sigma\)-algebra over \(X\). (2 pts)

(b) Show that \(D = \sigma(\mathcal{C})\). (1 pt)

3. Let \((X, \mathcal{A}, \mu)\) be a finite measure space (so \(\mu(X) < \infty\)), and \(T : X \to X\) an \(\mathcal{A}/\mathcal{A}\)-measurable function satisfying \(\mu(A) = \mu(T^{-1}(A))\) for all \(A \in \mathcal{A}\). For \(n \geq 1\), denote by \(T^n = T \circ T \circ \cdots \circ T\) the \(n\)-fold composition of \(T\) with itself.

(a) For \(B \in \mathcal{A}\), let \(D(B) = \{x \in B : T^n(x) \notin B \text{ for all } n \geq 1\}\). Show that \(D(B) \in \mathcal{A}\). (1 pt)

(b) For \(n \geq 1\), let \(D(B)_n = T^{-n}(D(B))\). Show that \(\mu(D(B)_n) = \mu(D(B))\), for \(n \geq 1\), and that \(D(B)_n \cap D(B)_m = \emptyset\) if \(n \neq m\). (1 pt)

(c) Show that \(\mu(D(B)) = 0\). (1 pt)

(d) Suppose \(A \in \mathcal{A}\) satisfies the property that if \(B \in \mathcal{A}\) with \(\mu(B) > 0\), then there exists \(n \geq 1\) such that \(\mu(A \cap T^{-n}B) > 0\). Show that \(\mu(A) > 0\), and if additionally \(T^{-1}(A) = A\), then \(\mu(A) = \mu(X)\). (1 pt)