Full retake exam Econometrics (WISB377)

Thursday, 2 February 2023, 11:00 – 13:00 CET full retake

*The students who were given extra time from the Board of Examiners have additional 20 minutes (end time of the exam: 13:40 CET).

Remarks:
- This entrance test consists of 12 sub-questions 6 numbered pages (included front page).
- Write your name and registration number on each page of your exam.
- Please do not post copies of this exam on the Internet.
**Question 1**

a) The Ordinary Least Squares estimator (OLS estimator) of $\beta$ is obtained by

$$\hat{\beta} = \arg\min_\beta L(\beta)$$

for which the loss function is

$$L(\beta) = (y - X\beta)'(y - X\beta)$$

**Question 1.** please derive the OLS estimator $\hat{\beta}$. What are the necessary assumptions?

**Question 2.** why is the Hessian of this minimization procedure a positive definite matrix?

b) For a set of information of $n$ firms, consider the following population regression equation.

$$\log(C\text{osts}_i) = \beta_0 + \beta_1 \log(\text{Firmsize}_i) + \beta_2 \text{Productivity}_i + \beta_3 \text{DumOld}_i + u_i \quad i=1,\ldots,n$$

log is the natural logarithm, Costs is the costs of a firm in thousands of euros, Firmsize is the number of employees and Productivity is the value of the production per worker in thousands of Euros. DumOld is a 0-1 indicator variable that has the value of 1 if the firm already existed prior to the year 2000 (and zero elsewhere).

**Question:** please give a precise interpretation of the regression parameters $\beta_1, \beta_2$ and $\beta_3$

c) For the bivariate regression equation

$$y_i = \beta_0 + \beta_1 x_i + u_i \quad i = 1,\ldots,n$$

and a random sample of $n$ observations, it is assumed that $E(u | X) = 0$ (strict exogeneity), for which $X$ is a $(n \times 2)$-dimensional matrix.

**Question:** Using the Law of Iterated Expectations, please proof that $E(u | X) = 0 \Rightarrow Cov(u, x) = 0$.

d) We formulate $\beta = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}$. Let’s continue with equation (4) in the previous exercise. The $n$ vectors $\begin{pmatrix} 1 \\ x_1 \end{pmatrix}, \ldots, \begin{pmatrix} 1 \\ x_n \end{pmatrix}$ are identically and independently distributed 2-dimensional
random variables for which \( E\left( \begin{pmatrix} 1 \\ x_i \end{pmatrix} \right) = C \) where \( C \) is a finite and non-singular matrix. In addition, the \( n \) random variables \( x_1u_1, x_2u_2, \ldots, x_nu_n \), are identically and independently distributed with \( Ex_iu_i = 0 \). Furthermore, \( u_1, u_2, \ldots, u_n \), are identically and independently distributed with \( Eu_i = 0 \). We consider the OLS estimator \( \hat{\beta}_n \).

**Question:** demonstrate how you can make use of all of these assumptions to proof that the OLS-estimator converges in probability to \( \beta \)

\[ \hat{\beta}_n \xrightarrow{p} \beta \quad \text{as} \quad n \to \infty \]

**Question 2**

For a random sample of \( n \) observations, we consider the 4-dimensional vector of regression parameters \( \beta \) of the linear regression model

\[ y = X\beta + u \]

\( \beta \) is estimated by the OLS estimator. It is assumed that the column rank of \( X \) is 4, and that the conditional distribution \( u \mid X \sim \text{Normal}(0, I_n) \) for which \( I_n \) is an identity matrix of dimension \( n \), and \( \mathbf{0} \) an \( n \)-dimensional vector of zeros.

A researcher formulates a null hypothesis: \( \beta_1 = 0 \) and \( \beta_2 = \beta_3 \).

**Question:** Please show for which matrix \( R \) and vector \( r \) the null hypothesis can be written as

\[ R\beta = r \]

a) **Question:** Please derive the Wald test statistic under the null hypothesis, \( R\beta = r \), and explain how the aforementioned assumptions are required for the derivation.

b) **Question:** What is the statistical distribution of the test statistic under the null hypothesis?
Question 3
For a sample of \( n \) observations, for the linear regression model:

\[
y = X\beta + u
\]

let’s assume that the variance covariance matrix of the error terms contains heteroskedasticity:

\[
Var(u \mid X) = \text{diag}(\sigma_i^2) \quad i = 1, \ldots, n
\]

a) Please, discuss the consequences of heteroskedasticity for the consistency of the OLS estimator

b) It can be demonstrated that \( Var(\hat{\beta} \mid X) = (X'X)^{-1}X'\Psi X(X'X)^{-1} \), for which

\[
X'\Psi X = \sum_{i=1}^{n} x_i x_i', \sigma_i^2.
\]

Please carefully describe both stages of the estimation procedure to calculate the White robust standard errors.
Question 4

Let’s consider the moving average model. The MA(1) model is:

\[ u_t = e_t + \alpha e_{t-1} \quad t = 2, ..., T \]

Subscript \( t \) refers to the \( t \)-th period. The error term \( e_t \) is i.i.d. (identically and independently distributed), with expected value of zero and constant variance:

\[ Ee_t = 0 \text{ and } Var(e_t) = \sigma_e^2. \]

**Question:** Show that the \((T-1) \times (T-1)\) covariance matrix of the error term \( u \)

\[
Var(u \mid X) = \Psi = \\
\begin{pmatrix}
(1 + \alpha_i^2)\sigma_e^2 & \alpha_i\sigma_e^2 & 0 & \cdots & 0 \\
\alpha_i\sigma_e^2 & (1 + \alpha_i^2)\sigma_e^2 & \alpha_i\sigma_e^2 & 0 \\
0 & \alpha_i\sigma_e^2 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\alpha_i\sigma_e^2 & \alpha_i\sigma_e^2 & \cdots & (1 + \alpha_i^2)\sigma_e^2 & \alpha_i\sigma_e^2 \\
0 & \cdots & 0 & \alpha_i\sigma_e^2 & (1 + \alpha_i^2)\sigma_e^2
\end{pmatrix}
\]
**Question 5**

We consider the panel data model

$$y_{it} = x_{it}' \beta + \alpha_i + u_{it} \quad i = 1, \ldots, n; t = 1, \ldots, T$$

for which $\alpha_i$ is the individual-specific effect (random variable), and $u_{it}$ is the identically and independently distributed error term with expected value zero and constant variance.

a) **Question**: Please calculate the autocorrelation for the random effects estimator. What are the major assumptions of the random effects estimator?

b) **Question**: For the model at the level of the individual

$$y_i = \alpha_i \mathbf{1} + X_i \beta + u_i,$$

$$y_i = \begin{pmatrix} y_{i1} \\ y_{i2} \\ \vdots \\ y_{iT} \end{pmatrix}; \quad X_i = \begin{pmatrix} x_{i11} & x_{i12} & \cdots & x_{i1k} \\ x_{i21} & x_{i22} & \cdots & x_{i2k} \\ \vdots & \vdots & \ddots & \vdots \\ x_{i(T-1)1} & x_{i(T-1)2} & \cdots & x_{iT} \end{pmatrix}; \quad u_i = \begin{pmatrix} u_{i1} \\ u_{i2} \\ \vdots \\ u_{iT} \end{pmatrix}$$

$y_i$ and $u_i$: $T \times 1$ vectors for individual $i$; $X_i$: $T \times k$ matrix for individual $i$; $\mathbf{1}$ is a $T \times 1$ vector of ones.

We want to derive the first-difference estimator. Show how we can make use of the $((T-1) \times T)$-matrix $D$ to obtain the first-difference estimator of $\beta$.

$$D = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

c) **Question**: What are the essential assumptions for the first-difference estimator? Using the derivation of the previous sub-question, please give a careful motivation for your answer.