Utrecht University
Mathematical Institute

Final Exam for Introduction to Financial Mathematics, WISB373

Monday January 30 2023, 13:30-16:30 o'clock **(3 hours examination)**

1. Answer the statements below by either TRUE or FALSE. In case of "TRUE", explain your answer. In case of "FALSE", explain why the statement is false and give the correct statement. (total 10 points)

a. A standard European option gives the option holder the right to buy or sell an option from a writer at time $t = T$.

b. The put-call parity is a relation between call and put option prices that should hold at any time $t \in [0, T]$, with T the expiry date.

c. The Black-Scholes price is the fair price of a European option when the stock price is modeled by a Geometric Brownian Motion.

d. A European option is only traded on European exchanges, while an American option is only traded in North-, South- and Central America.

2. Let $W(t)$ be a standard Brownian motion. Evaluate

$$\int_0^T W^2(t)\mathrm{d}t + \int_0^T 2\,t\,W(t)\mathrm{d}W(t).$$

Which of the answers below is correct,

I) 0,

II) $W^3(T)/3 + T^2W^2(T) - T^2/2$,

III) $2W^3(T)/3 + T^2W(T) - T^2$,

IV) $TW^2(T) - T^2/2$,

V) $2TW^2(T) - T$.

Justify your answer (10 pts.).

3. Let $\theta \in [0, 2\pi)$ and $W = (W_1, W_2)$ be a two-dimensional Brownian motion. Use Lévy's characterisation of Brownian motion to show that if

$$Y_1(t) = \cos(\theta)W_1(t) - \sin(\theta)W_2(t), \quad Y_2(t) = \sin(\theta)W_1(t) + \cos(\theta)W_2(t)$$

then $Y = (Y_1, Y_2)$ is a two-dimensional Brownian motion. (10 pts.)

4. Let $\{W(t) : t \geq 0\}$ be a Brownian motion, and define the process $\{X(t) : t \geq 0\}$ as

$$X(t) = W^3(t) + \ c\,t\,W(t).$$

For what value of c is $X(t)$ a martingale? Find a process $\{\Gamma(t), t \geq 0\}$ adapted to $\{\mathcal{F}(t), t \geq 0\}$ such that

$$X(t) = X(0) + \int_0^t \Gamma(s)\mathrm{d}W(s).$$

(10 pts.)

5. Theorem: Let $\{X_1(t), t \geq 0\}$ and $\{X_2(t), t \geq 0\}$ be the diffusion processes

$$\mathrm{d}X_i(t) = \alpha_i(t)\mathrm{d}t + \sigma_i(t)\mathrm{d}W(t), \quad i = 1, 2.$$

Then $\{X_1(t)X_2(t), t \geq 0\}$ is the diffusion process given by

$$\mathrm{d}(X_1(t)X_2(t)) = X_2(t)\mathrm{d}X_1(t) + X_1(t)\mathrm{d}X_2(t) + \sigma_1(t)\sigma_2(t)dt.$$

Prove the theorem in the case that α_i and σ_i are deterministic constants and $X_i(0) = 0$, for $i = 1, 2$. (10 pts.)

6. Let $\{W(t) : 0 \leq t \leq T\}$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with $\{\mathcal{F}(t) : 0 \leq t \leq T\}$ its natural filtration, and $\mathcal{F} = \mathcal{F}(T)$. Consider a stock price process $\{S(t) : 0 \leq t \leq T\}$ with $S(t) = t^3 + 3W(t)$.

a. (10 pts.) Construct a measure $\tilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ (i.e., $\tilde{\mathbb{P}}(A) = 0$ if and only if $\mathbb{P}(A) = 0,\ A \in \mathcal{F}$), such that the price process $\{S(t) : 0 \leq t \leq T\}$ is a martingale under $\tilde{\mathbb{P}}$ and with respect to the filtration $\{\mathcal{F}(t) : 0 \leq t \leq T\}$.

b. (10 pts.) Consider a European call option on this stock with expiration date T and strike price K. Find an expression for the price of this option at time 0, with constant interest rate r:

$$C(0) = \tilde{\mathbb{E}}[\mathrm{e}^{-rT}(S(T) - K)^+].$$

7. (10 pts.) The price of a call option under Geometric Brownian Motion stock dynamics, with exercise price K and expiry date T, time t and stock price $S(t)$ is given by:

$$c(S, t) = SN(d_+) - Ke^{-r(T-t)}N(d_-),$$

where

$$d_\pm = \frac{\log(S/K) + (r \pm \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}},$$

and

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy$$

is the $N(0, 1)$ CDF. Use the put-call parity to show that the corresponding put option value is given by:

$$p(S, t) = Ke^{-r(T-t)}N(-d_-) - SN(-d_+).$$

8. Answer the statements below by either TRUE or FALSE. In case of "TRUE", explain your answer. In case of "FALSE", explain why the statement is false and give the correct statement. (total 10 pts.)

a. A market model is complete if there exists a risk-neutral measure under which the discounted asset price is a martingale.

b. A market model does not admit arbitrage if every risk-free portfolio grows at the same interest rate $R(t)$.

Please, make sure that your name is written down on each of the submitted solutions.