Utrecht University
Mathematical Institute

Answers Re-Examination for Introduction to Financial Mathematics, WISB373

Monday March 13th 2023, 17:00-20:00 o'clock **(3 hours examination)**

(Each item is worth 10 points)

1. Assume we have a European call $c(t)$ and a put option $p(t)$, with the same expiry date $T = 4$, i.e., exercise in 4 years, and strike price $K = 10$ Euro. The current share price is 11 Euro, assuming a zero constant interest rate $r = 0\%$. Determine if there exists an arbitrage opportunity if both options currently have the value $c(0) = 2.5$ Euro and $p(0) = 1.5$ Euro.

Answer 1. We form two portfolios using the options, the underlying asset and a cash amount K, with one based on the put $p(t)$ and the other based on the call $c(t)$, as follows,

$$\Pi_1(t) = p(t) + S(t),$$
$$\Pi_2(t) = c(t) + K.$$

(as $e^0 = 1$). These portfolios have same value at expiry time T. moreover, by the put-call parity, we see that their values are also equal any time prior to the exercise time, particularly at time $t = 0$. So, there is no arbitrage opportunity here!

2. **a.** The random process $Z(t)$ is defined as $Z(t) = \alpha W(t) - \sqrt{\beta} W^*(t)$, where $W(t)$ and $W^*(t)$ are independent standard Brownian motions. Determine the relationship between α and β for which $Z(t)$ is a Brownian motion.

b. Determine whether $W(t) + 4t$ is a martingale.

c. Let $v_1, v_2, v_3 \in \mathbb{R}^3$ be orthonormal vectors, i.e. $v_i \cdot v_j = \delta_{ij}$. If $W(t) = (W_1(t), W_2(t), W_3(t))$ is a three-dimensional Brownian motion and $X_j(t) = v_j \cdot W(t)$ for $j \in \{1, 2, 3\}$, show, with the help of Lévy's characterization, that (X_1, X_2, X_3) is another three-dimensional Brownian motion.

Answer 2a.: For the combination of Brownian Motions, we find:

$$\begin{aligned} \mathbb{E}[Z(t)] &= \mathbb{E}[\alpha W(t) - \sqrt{\beta} W^*(t)] \\ &= \alpha \mathbb{E}[W(t)] - \sqrt{\beta} \mathbb{E}[W^*(t)] \\ &= 0 \end{aligned}$$

For the variance, we find:

$$\begin{aligned} \mathbb{V}\mathrm{ar}[Z(t+u) - Z(t)] &= \mathbb{V}\mathrm{ar}[(\alpha W(t+u) - \sqrt{(\beta)} W^*(t+u)] \\ &\quad -(\alpha W(t) - \sqrt{\beta} W^*(t)] \\ &= \mathbb{V}\mathrm{ar}[(\alpha(W(t+u) - W(t) - \sqrt{(\beta)}(W^*(t+u) - W^*(t)] \\ &= \mathbb{V}\mathrm{ar}[(\alpha(W(t+u) - W(t)] - \mathbb{V}\mathrm{ar}[\sqrt{(\beta)}(W^*(t+u) - W^*(t)] \end{aligned}$$

because $W(t)$ and $W^*(t)$ are independent. So,

$$\mathbb{V}\mathrm{ar}[Z(t+u) - Z(t)] = \alpha^2 u + \beta u = (\alpha^2 + \beta) u.$$

This expression should equal u and not depend on t. It follows that: $\mathbb{V}\mathrm{ar}[Z(t+u) - Z(t)] = u$ if $\alpha^2 + \beta = 1$ or $\beta = 1 - \alpha^2$.

Answer 2b.: Let $s < t$. Substituting $W(t) = W(s) + (W(t) - W(s))$, gives

$$\begin{aligned}\mathbb{E}[W(t) + 4t|\mathcal{F}(s)] &= \mathbb{E}[W(s) + (W(t) - W(s)) + 4t|\mathcal{F}(s)] \\ &= \mathbb{E}[W(s)|\mathcal{F}(s)] + \mathbb{E}[W(t) - W(s)|\mathcal{F}(s)] + \mathbb{E}[4t|\mathcal{F}(s)] \\ &= W(s) + 0 + 4t\end{aligned}$$

So, $\mathbb{E}[W(t){+}4t|\mathcal{F}(s)] \neq W(s){+}4s$. Thus $W(t){+}4t$ is not a martingale. There was no need to decompose t.

Answer 2c.: We make use of Lévy's characterisation of Brownian motion: each X_j is a linear combination of continuous martingales (the entries W_j), hence a continuous martingale. Also $X_j(0) = 0$ for all j, since the underlying Brownian motions start at zero. Moreover,

$$\begin{aligned}[X_j, X_j](t) &= \left[\sum_{i=1}^{3} v_{ji}W_i(t), \sum_{i=1}^{3} v_{ji}W_i(t)\right] = \sum_{i,r=1}^{3} v_{ji}v_{jr}[W_i, W_r](t) \\ &= \sum_{i=1}^{3} v_{ji}^2 t = ||v_j||^2 t = t\end{aligned}$$

using the orthonormality property of the v_j. Further, for $j \neq \ell$,

$$\begin{aligned}[X_j, X_\ell](t) &= \sum_{i,r=1}^{3} v_{ji}v_{\ell,r}[W_i, W_r](t) \\ &= \sum_{i=1}^{3} v_{ji}v_{\ell i}t = v_j \cdot v_\ell = 0\end{aligned}$$

using the orthonormality property of the v_j.

3 Let $Q(t)$ denote the exchange rate at time t. It is the price in domestic currency of one unit of foreign currency and converts foreign currency into domestic currency. A model for the dynamics of the exchange rate is

$$\mathrm{d}Q(t)/Q(t) = \mu_Q \mathrm{d}t + \sigma_Q \mathrm{d}W(t).$$

This has the same structure as the common model for the stock price. The reverse exchange rate, denoted $R(t)$, is the price in foreign currency of one unit of domestic currency $R(t) = 1/Q(t)$. Derive $\mathrm{d}R(t)$

Answer 3: $R = 1/Q$ is a function of a single variable Q, so the Itô's-Doeblin formula says:

$$\mathrm{d}R = \frac{\mathrm{d}R}{\mathrm{d}Q}\mathrm{d}Q + \frac{1}{2}\frac{d^2R}{dQ^2}(\mathrm{d}Q)^2.$$

Substituting

$$\begin{aligned}\mathrm{d}Q &= Q[\mu_Q \mathrm{d}t + \sigma_Q \mathrm{d}W], \\ (\mathrm{d}Q)^2 &= Q^2\sigma_Q^2 \mathrm{d}t\end{aligned}$$

$$\frac{\mathrm{d}R}{\mathrm{d}Q} = \frac{-1}{Q^2}\frac{\mathrm{d}^2R}{\mathrm{d}Q^2} = \frac{2}{Q^3}.$$

gives

$$\begin{aligned}\mathrm{d}R &= \frac{-1}{Q^2}Q[\mu_Q \mathrm{d}t + \sigma_Q \mathrm{d}W] + \frac{1}{2}\frac{2}{Q^3}Q^2\sigma_Q^2 \mathrm{d}t \\ &= \frac{-1}{Q}[\mu_Q \mathrm{d}t + \sigma_Q \mathrm{d}W] + \frac{1}{Q}\sigma_Q^2 \mathrm{d}t \\ &= -R[\mu_Q \mathrm{d}t + \sigma_Q \mathrm{d}W] + R\sigma_Q^2 \mathrm{d}t \\ &= R[-\mu_Q + \sigma_Q^2]\mathrm{d}t - R\sigma_Q \mathrm{d}W.\end{aligned}$$

Dividing by $R(t) \neq 0$ gives the dynamics of $R(t)$:

$$\frac{\mathrm{d}R(t)}{R(t)} = (-\mu_Q + \sigma_Q^2)\mathrm{d}t - \sigma_Q \mathrm{d}W(t)$$

4. Let $\{W(t) : t \geq 0\}$ be a Brownian motion with filtration $\{\mathcal{F}(t) : t \geq 0\}$.
Let $Y(t) = \int_0^t W^2(u)\mathrm{d}W(u) - \frac{1}{2}\int_0^t W^4(u)\mathrm{d}u$ and $X(t) = e^{Y(t)}$, for $t \geq 0$.

a. Prove that $X(t) = 1 + \int_0^t X(u)W^2(u)\mathrm{d}W(u)$, for $t \geq 0$.

b. Prove that the process $\{X(t) : t \geq 0\}$ is a martingale with respect to the filtration $\{\mathcal{F}(t) : t \geq 0\}$. Show that $\mathbb{E}(X(t)) = 1$ and $\mathbb{V}ar[X(t)] = \int_0^t \mathbb{E}[W^4(u)X^2(u)]\mathrm{d}u$ for $t \geq 0$.

Answer 4a.: Note that $\{Y(t) : t \geq 0\}$ is an Itô process with $\mathrm{d}Y(t) = W^2(t)\mathrm{d}W(t) - \frac{1}{2}W^4(t)\mathrm{d}t$ and $\mathrm{d}Y(t)\mathrm{d}Y(t) = W^4(t)\mathrm{d}t$. Using the Itô-Doeblin formula for Itô processes with $f(x) = \mathrm{e}^x$, we get

$$\begin{aligned}
X(t) &= f(Y(t)) \\
&= f(Y(0)) + \int_0^t X(u)\mathrm{d}Y(u) + \frac{1}{2}\int_0^t X(u)\mathrm{d}Y(u)\mathrm{d}Y(u) \\
&= 1 + \int_0^t X(u)W^2(u)\mathrm{d}W(u) - \frac{1}{2}\int_0^t X(u)W^4(u)\mathrm{d}u + \frac{1}{2}\int_0^t X(u)W^4(u)\mathrm{d}u \\
&= 1 + \int_0^t X(u)W^2(u)\mathrm{d}W(u).
\end{aligned}$$

Answer 4b.: First note that the process $\{Y(t) : t \geq 0\}$ is an Itô process. Since the Itôˆintegral $\{\int_0^t X(u)W^2(u)\mathrm{d}W(u) : t \geq 0\}$ seen as a process is a martingale, we conclude that the process $\{X(t) : t \geq 0\}$ is a martingale with respect to the filtration $\{\mathcal{F}(t) : t \geq 0\}$. Thus, $\mathbb{E}[X(t)] = \mathbb{E}[X(0)] = 1$. To calculate the variance, we use the Itô-isometry and Fubini's Theorem (to interchange the integral with the expectation) to get

$$\begin{aligned}
\mathbb{V}ar[X(t)] &= \mathbb{E}[(X(t) - 1)^2] \\
&= \mathbb{E}\left[\left(\int_0^t X(u)W^2(u)\mathrm{d}W(u)\right)^2\right] \\
&= \mathbb{E}\left[\int_0^t X^2(u)W^4(u)\mathrm{d}u\right] \\
&= \int_0^t \mathbb{E}[X^2(u)W^4(u)]\mathrm{d}u.
\end{aligned}$$

5. Given a Radon-Nikodym derivative Z, and the associated Radon-Nikodym process $\{Z(t) : t \geq 0\}$, defined by $Z(t) = \mathbb{E}[Z|\mathcal{F}(t)]$, where $\{\mathcal{F}(t) : t \geq 0\}$ is a given filtration. We then have the change of probability measure, $\mathrm{d}\tilde{\mathbb{P}} = Z\mathrm{d}\mathbb{P}$, with the expectation under the $\tilde{\mathbb{P}}$-measure, i.e., $\tilde{\mathbb{E}}[Y] = \mathbb{E}[ZY]$

Let Y be a random variable which is $\mathcal{F}(t)$-measurable. Prove that

$$\mathbb{E}[YZ] = \tilde{\mathbb{E}}[Y] = \mathbb{E}[YZ(t)].$$

Suppose Y is $\mathcal{F}(t)$-measurable, then prove (using partial averaging) that, for $s < t$

$$\tilde{\mathbb{E}}[Y|\mathcal{F}(s)] = \frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}(s)].$$

Answer 5.: We look for the proof of Lemma 5.2.1 from the book. Recall: $\tilde{\mathbb{E}}[Y] = \mathbb{E}[ZY]$, Y is a r.v. so Y is $\mathcal{F}(t)$-measurable.

Let $\{\mathcal{F}(t) : t \geq 0\}$ be a given filtration (for which we have defined the Radon-Nikodym process).

We consider the RHS

$$\mathbb{E}[YZ(t)] = \mathbb{E}[Y\mathbb{E}[Z|\mathcal{F}(t)] = \mathbb{E}[\mathbb{E}[YZ|\mathcal{F}(t)]] = \mathbb{E}[YZ] = \tilde{\mathbb{E}}[Y]$$

using the $\mathcal{F}(t)$ measurability of Y.

Here we look for the proof of book's Lemma 5.2.2:

Recall: $\tilde{\mathbb{E}}[Y] = \mathbb{E}[ZY]$, Y is a r.v., so Y is $\mathcal{F}(t)$-measurable. To prove the result, it is enough to show that the RHS is the conditional expectation of Y given $\mathcal{F}(s)$ under the measure $\tilde{\mathbb{P}}$.

So, we need to verify the two defining conditions of conditional expectations.

(i) Clearly the RHS is $\mathcal{F}(s)$-measurable. $Z(s)^{-1}$ is $\mathcal{F}(s)$-measurable; the same holds for the second term. Hence, the product is $\mathcal{F}(s)$-measurable.

(ii) Now, let $A \in \mathcal{F}(s)$, we want to show

$$\begin{aligned}
\int_A \frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}(s)]\mathrm{d}\tilde{\mathbb{P}} &= \int_A Y\mathrm{d}\tilde{\mathbb{P}} = \tilde{\mathbb{E}}[\mathbb{1}_A Y] \\
\int_A \frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}(s)]\mathrm{d}\tilde{\mathbb{P}} &= \tilde{\mathbb{E}}[\mathbb{1}_A \frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}(s)]] \\
&= \tilde{\mathbb{E}}[\mathbb{E}[\mathbb{1}_A \frac{1}{Z(s)} YZ(t)|\mathcal{F}(s)]] \text{ use Lemma 5.2.1} \\
&= \mathbb{E}[Z(s)\mathbb{E}[\mathbb{1}_A \frac{1}{Z(s)} YZ(t)|\mathcal{F}(s)]] \\
&= \mathbb{E}[\mathbb{E}[\mathbb{1}_A YZ(t)|\mathcal{F}(s)]] = \mathbb{E}[\mathbb{1}_A YZ(t)] \\
\text{(using again Lemma 5.2.1)} &= \tilde{\mathbb{E}}[1_A Y] = \int_A Y\mathrm{d}\tilde{\mathbb{P}} \\
\text{So, } \frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}(s)] &= \tilde{\mathbb{E}}[Y|\mathcal{F}(s)]
\end{aligned}$$

Let $\{W(t) : 0 \leq t \leq T\}$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\{\mathcal{F}(t) : 0 \leq t \leq T\}$ be its natural filtration, and assume $\mathcal{F} = \mathcal{F}(T)$. Consider a stock with price process $\{S(t) : 0 \leq t \leq T\}$ with

$$S(t) = S(0)\exp\left\{\int_0^t \mathrm{e}^{-u}\mathrm{d}W(u) + \int_0^t (1 - \frac{1}{2}\mathrm{e}^{-2u})\mathrm{d}u\right\}$$

a. Let

$$X(t) = \int_0^t \mathrm{e}^{-u}\mathrm{d}W(u) + \int_0^t (1 - \frac{1}{2}\mathrm{e}^{-2u})\mathrm{d}u$$

Determine the distribution of $X(t)$.

b. Prove the $\{S(t) : t \geq 0\}$ is an Itô process.

Answer 6a. Let $Y(t) = \int_0^t \mathrm{e}^{-u}\mathrm{d}W(u)$. Since $Y(t)$ is the Itô integral of a deterministic process, by Theorem 4.4.9 $Y(t)$ is normally distributed with $\mathbb{E}[Y(t)] = 0$ and $\mathbb{V}ar[Y(t)] = \int_0^t \mathrm{e}^{-2u}\mathrm{d}u = \frac{1}{2}(1 - \mathrm{e}^{-2t})$. Since $X(t) = Y(t) + \int_0^t (1 - \frac{1}{2}e^{-u})\mathrm{d}u = Y(t) + t + \frac{1}{4}(\mathrm{e}^{-2t} - 1)$, we see that $X(t)$ is normally distributed with mean $\mathbb{E}[X(t)] = t + \frac{1}{4}(\mathrm{e}^{-2t} - 1)$ and variance $\mathbb{V}ar[X(t)] = \mathbb{V}ar[Y(t)] = \frac{1}{2}(1 - \mathrm{e}^{2t})$.

Answer 6b. With $X(t) = \int_0^t \mathrm{e}^{-u}\mathrm{d}W(u) + \int_0^t (1-\frac{1}{2}\mathrm{e}^{-2u})\mathrm{d}u$ we have $\mathrm{d}X(t) = \mathrm{e}^{-t}\mathrm{d}W(t) + (1-\frac{1}{2}\mathrm{e}^{-2t})\mathrm{d}t$ and $\mathrm{d}X(t)\mathrm{d}X(t) = \mathrm{e}^{-2t}\mathrm{d}t$. Note that $S(t) = S(0)\mathrm{e}^{X(t)}$, so let $f(x) = S(0)\mathrm{e}^{x}$, then $f_x(x) = f_{xx}(x) = f(x)$. By the Itô-Doeblin formula, we have,

$$
\begin{aligned}
\mathrm{d}S(t) &= (X(t)) = S(t)\mathrm{d}X(t) + \frac{1}{2}S(t)\mathrm{d}X(t)\mathrm{d}X(t) \\
&= S(t)\left(\mathrm{e}^{-t}\mathrm{d}W(t) + (1-\frac{1}{2}\mathrm{e}^{-2t})\mathrm{d}t\right) + \frac{1}{2}S(t)\mathrm{e}^{-2t}\mathrm{d}t \\
&= S(t)\mathrm{d}t + S(t)\mathrm{e}^{-t}\mathrm{d}W(t).
\end{aligned}
$$

This shows that $S(t) = S(0) + \int_0^t S(u)\mathrm{d}u + \int_0^t S(u)\mathrm{e}^{-u}\mathrm{d}W(u)$. Hence, $\{S(t) : t \geq 0\}$ is an Itô process.