Universiteit Utrecht Boedapestlaan 6
Mathematisch Instituut 3584 CD Utrecht

Stochastic Processes: Final, 2022-23
You are allowed to use the *Summary of Important Results Parts 1 and 2*

(1) Consider the discrete Markov chain $(X_n)_{n\geq 0}$ with state space $I = \{1,2,3,4\}$ and transition matrix

$$P = \begin{bmatrix} 0.8 & 0.2 & 0 & 0 \\ 0.6 & 0.2 & 0 & 0.2 \\ 0 & 0 & 0.2 & 0.8 \\ 0 & 0.4 & 0.6 & 0 \end{bmatrix}.$$

(a) Prove that the Markov chain is irreducible and aperiodic. (0.5 pt)

(b) Find the stationary distribution π and show that if $(X_n)_{n\geq 0}$ is Markov(π, P), then it is time reversible. (1 pt)

(c) What is the frequency that the Markov chain is in state $i = 1,2,3,4$. (0.5 pt)

(2) Assume that passengers arrive at a bus station according to a Poisson process with rate $\lambda = 2$ per minute. Suppose that the passengers independently come in two types: woman and man. The probability of a woman arriving is $\frac{2}{5}$ and the probability of a man arriving is $\frac{3}{5}$.

(a) Find the probability that the twelfth passenger arrives at least two minutes after the tenth passenger. (1 pt)

(b) Find the probability that three passengers arrive in the time interval $[1,5)$ and two passengers arrive in $[2,6)$. (1 pt)

(c) Suppose 12 passengers arrived in the interval $[0,5]$, what is the probability that 6 of those are women? (1 pt)

(d) Find the probability that in the interval $[0,5]$, 6 women and 3 men arrive. (1 pt)

(3) The number of goals in a hockey game is modelled as a Poisson process with rate $\frac{1}{15}$ goal per minute.

(a) In a 60 minutes game, find the probability that the second goal occurs in the first 15 minutes of the game. (1 pt)

(b) Suppose we are told that in a 60 minutes game at least 2 goals occurred in the first 15 minutes. Determine the (conditional) probability that a total of 4 goals were scored in the game. (1 pt)

(c) Suppose in a 60 minutes game we are told that in the first 15 minutes only one goal was scored, what is the (conditional) probability that the first goal was scored in the first 5 minutes? (1 pt)

(4) Consider the Q-matrix

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Denote the state space by $I = \{1,2,3\}$. Determine an explicit expression for the corresponding stochastic matrix $P(t) = e^{tQ}$. (1 pt)