Stochastic Processes: Mid-Term, 2022-23

(1) Consider the Markov chain \((X_n)_{n\geq 0}\) with state space \(I = \{1, 2, 3, 4\}\) and transition matrix
\[
\begin{bmatrix}
\frac{1}{4} & \frac{3}{4} & 0 & 0 \\
\frac{1}{3} & 0 & 0 & \frac{2}{3} \\
0 & 0 & \frac{1}{2} & \frac{1}{2} \\
0 & 0 & 1 & 0
\end{bmatrix}.
\]

(a) Determine the communicating classes, which ones are recurrent and which ones are transient. Justify your answer. (1.5 pts)

(b) For \(i = 1, 2\), determine the value of \(\mathbb{P}_i(T_i = \infty)\), where \(T_i = \inf\{n \geq 1 : X_n = i\}\) is the first passage time to state \(i\). (Hint: \(\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}\), for \(|x| < 1\).) (1.5 pts)

(c) Determine \(E_i[T_i]\) for \(i = 1, 4\). (Hint: \(\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}\), for \(|x| < 1\).) (1.5 pts)

(2) Consider a Markov chain \((X_n)_{n\geq 0}\) with state space \(I = \{0, 1, 2, \ldots\}\) and transition probabilities
\[
p_{i,i+1} = \frac{i+1}{i+2}, \quad p_{i,0} = \frac{1}{i+2}, \quad i \geq 0.
\]
Prove that the Markov chain is irreducible and null-recurrent. (Hint: \(\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1\)) (2 pts)

(3) Consider a Markov chain \((X_n)_{n\geq 0}\) with state space \(\{1, 2, 3\}\) and transition matrix
\[
P = \begin{bmatrix}
\frac{3}{4} & \frac{1}{4} & 0 \\
\frac{1}{5} & \frac{3}{4} & \frac{1}{5} \\
0 & \frac{1}{3} & \frac{2}{3}
\end{bmatrix}.
\]
The eigenvalues of \(P\) are given by \(\theta_1 = 1\), \(\theta_2 = \frac{1}{2}\) and \(\theta_3 = \frac{3}{4}\).

(a) Prove that \(p_{11}^{(n)} = \frac{1}{4} + \frac{1}{4}(\frac{3}{4})^n + \frac{1}{2}(\frac{3}{4})^n\), \(n \geq 1\). (2 pts)

(b) Determine the value of \(E_1[T_1]\), where \(T_1 = \inf\{n \geq 1 : X_n = 1\}\) is the first passage time to state \(1\). (1.5 pts)