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Stochastic Processes: Mid-Term, 2022-23

(1) Consider the Markov chain $(X_n)_{n\geq 0}$ with state space $I = \{1, 2, 3, 4\}$ and transition matrix

$$\begin{bmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

(a) Determine the communicating classes, which ones are recurrent and which ones are transient. Justify your answer. (1.5 pts)

(b) For $i = 1, 2$, determine the value of $\mathbb{P}_i(T_i = \infty)$, where $T_i = \inf\{n \geq 1 : X_n = i\}$ is the first passage time to state i. (Hint: $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, for $|x| < 1$.) (1.5 pts)

(c) Determine $\mathbb{E}_i[T_i]$ for $i = 1, 4$. (Hint: $\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$, for $|x| < 1$.) (1.5 pts)

(2) Consider a Markov chain $(X_n)_{n\geq 0}$ with state space $I = \{0, 1, 2, \ldots\}$ and transition probabilities

$$p_{i,i+1} = \frac{i+1}{i+2},\ p_{i,0} = \frac{1}{i+2},\ i \geq 0.$$

Prove that the Markov chain is irreducible and null-recurrent. (Hint: $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$) (2 pts)

(3) Consider a Markov chain $(X_n)_{n\geq 0}$ with state space $\{1, 2, 3\}$ and transition matrix

$$P = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{8} & \frac{3}{4} & \frac{1}{8} \\ 0 & \frac{1}{4} & \frac{3}{4} \end{bmatrix}.$$

The eigenvalues of P are given by $\theta_1 = 1$, $\theta_2 = \frac{1}{2}$ and $\theta_3 = \frac{3}{4}$.

(a) Prove that $p_{11}^{(n)} = \frac{1}{4} + \frac{1}{4}(\frac{1}{2})^n + \frac{1}{2}(\frac{3}{4})^n$, $n \geq 1$. (2 pts)

(b) Determine the value of $\mathbb{E}_1[T_1]$, where $T_1 = \inf\{n \geq 1 : X_n = 1\}$ is the first passage time to state 1. (1.5 pts)