<u>Final exam, Mathematical Modelling</u> (WISB357)

Wednesday, 9 Nov 2022, 13:30-16:30, Olympos Hal 3

- You may use one A4 sheet with hand-written notes (front and back) while working the problems.
- Write your name on each page you turn in, and additionally, on the first page, write your student number and the *total number of pages submitted.*
- For each question, motivation your answer.
- You may make use of results from previous subproblems, even if you have been unable to prove them.
- The maximum number of points per subproblem are given in italics between square brackets.
- Your grade is the total earned points divided by 3.
- The final exam weighs 60% in your grade for the course.

<u>Problem 1</u>. [*Reaction equations; stability; model interpretation.*] The following model is the standard SIR model with 'vital dynamics'

$$\frac{dS}{dt} = m(I + R) - \beta IS,$$
$$\frac{dI}{dt} = \beta IS - (m + g)I,$$
$$\frac{dR}{dt} = gI - mR,$$

with initial conditions $S(0) = S_0 > 0$, $I(0) = I_0 > 0$, $R(0) = 0$.

(a) [2pts] Use a conservation law to reduce the system to a problem for only S and I.

(b) [2pts] Explain why the solution must satisfy $0 \leq S(t) \leq N$, $0 \leq I(t) \leq N$, where $N = S_0 + I_0$.

(c) [2pts] What are the steady states, and what assumptions (if any) are needed to satisfy the conditions of part (b).

(d) [2pts] For one of the steady states, $I^* = 0$. Under what conditions is this equilibrium stable?

(e) [2pts] For one of the steady states, $I^* \neq 0$. Under what conditions is this equilibrium stable?

<u>Problem 2</u>. [*Nonlinear conservation laws.*] Consider the Burgers equation, a hyperbolic conservation law,

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial \rho}{\partial x} = 0, \qquad -\infty < x < \infty, \quad t > 0,$$

with initial condition

$$\rho(x, 0) = \rho_0(x) = \begin{cases} \rho_L, & x \leq 0 \\ \rho_R, & x > 0. \end{cases}$$

(a) [2pt] What is the flux function $J(\rho)$ for this equation? What is the wave speed $c(\rho)$?

(b) [2pt] What is the Rankine-Hugoniot condition that determines the location of a shock wave for this wave function?

(c) [2pt] Suppose $\rho_L = 2$ and $\rho_R = 1$. Give the solution $\rho(x,t)$ and make a sketch of the characteristics.

(d) [4pt] Suppose $\rho_R = 2$ and $\rho_L = 1$. Give the solution $\rho(x,t)$ and make a sketch of the characteristics. (Hint: consider an initial condition $\rho(x,0)$ that varies linearly between ρ_L and ρ_R over an interval $-\varepsilon \le x < \varepsilon$; determine the solution for this initial condition; and examine the limit $\varepsilon \to 0$.)

Problem 3. [*Continuum mechanics; asymptotic expansions.*] The momentum equation from continuum mechanics in material coordinates is

$$R_0(A)\frac{\partial^2 U}{\partial t^2}(A,t) = R_0(A)F(A,t) + \frac{\partial T}{\partial A}(A,t), \qquad 0 < A < \ell_0, \quad t > 0,$$

where $R_0(A) > 0$ is the density in material coordinates, $U(A,t)$ is the displacement of a cross-section A of air, $F(A,t)$ is the net external body force, and $T(A,t)$ is the stress.

Consider the steady state relation for a bungee cord of undeformed length ℓ_0, affixed at $A = 0$, hanging freely with no load, i.e. boundary conditions

$$U(0) = 0, \qquad T(\ell_0) = 0.$$

Suppose the density is uniform (R_0 is independent of A) and the external body force is constant $F(A) = g > 0$ (gravity).

(a) [2pts] Show that the steady-state stress $T(A)$ satisfies

$$T(A) = R_0 g(\ell_0 - A).$$

(b) [3pts] Suppose the bungee cord is nonlinear the the stress $T(A)$ related to the strain $\partial U/\partial A$ by

$$T(A) = E\frac{\partial U}{\partial A} + K\left(\frac{\partial U}{\partial A}\right)^3,$$

where $E > 0$ is the Young's modulus and $K > 0$ is a small constant. Nondimensionalize the problem, taking $U = \bar{U}u$, and $A = \bar{A}a$, where $\bar{U}$ and $\bar{A}$ are dimensional constants and u and a are dimensionless. Show that the resulting problem has the form

$$\frac{\partial u}{\partial a} + \varepsilon\left(\frac{\partial u}{\partial a}\right)^3 = 1 - a, \qquad 0 < a < 1,$$

where $u(0) = 0$, and ε is a small constant.

(c) [3pts] Find the the first two terms in the asymptotic expansion of u for small ε.

(d) [2pts] Convert your answer from part (c) back to the dimensional quantity $U(\ell_0)$ and estimate the deformed steady-state length of the bungee cord.