Instructions

- The exam is closed-book.
- Write your name and student number in all the pages of the exam.
- You may write your solutions in either Dutch or English.
- You must justify the claims you make.
- You may use results from the lectures or the dictaat, but you must provide a clear statement (with complete hypothesis and conclusion).
- Try to write with clear handwriting. Structure your explanations clearly, using one paragraph for each new idea and one sentence for each particular claim.
- Advice: Read all the exam in the beginning and address first the questions that you find easier.

Questions

Exercise 1 (1 point). Find a space $A$ and a subspace $B \subseteq A$ such that:

- $B$ is a retract of $A$,
- $B$ is not contractible,
- $B$ is not a deformation retract of $A$.

Exercise 2 (1.5 points). Let $X$ be a space. We define its category of opens $\mathcal{O}(X)$ as follows:

- Objects in $\mathcal{O}(X)$ are open subsets $U \subseteq X$.
- For each pair of objects $U \subseteq V$ in $\mathcal{O}(X)$, $\text{Hom}(U, V)$ contains a single element, the inclusion $i_{UV} : U \to V$. Otherwise, if $U$ is not a subset of $V$, $\text{Hom}(U, V)$ is empty.

Then:

- Verify that $\mathcal{O}(X)$ is a category.
- Prove that $U \cap V$ is the product of $U$ and $V$, as elements of $\mathcal{O}(X)$.
• Prove that $U \cup V$ is the coproduct of $U$ and $V$, as elements of $\mathcal{O}(X)$.

**Exercise 3** (1 point). Let $\gamma_k : S^1 \to S^1$ be the map $\gamma_k(z) = z^k$. Prove that the pushforward

$$(\gamma_k)_* : \Pi_1(S^1) \to \Pi_1(S^1)$$

is a groupoid isomorphism if and only if $k = \pm 1$.

**Exercise 4** (1 point). Construct a 2-dimensional cell complex, homotopy equivalent to the 2-torus, but which is not a surface.

**Exercise 5** (2 points). Let $f : S^1 \to T^2 = S^1 \times S^1$ be the map $z \mapsto (z, z^2)$ and $g : S^1 \to S^2$ the inclusion of the equator. Consider the space

$$X := \text{pushout}(T^2 \leftarrow S^1 \rightarrow S^2).$$

• Endow $X$ with the structure of a 2-dimensional cell-complex.

• Compute the fundamental group of $X$.

• Compute the first homologies of $X$.

**Exercise 6** (3.5 points). Consider the space $A := \mathbb{R}P^2 \vee T^2$. Fix a basepoint $a$.

• Endow $A$ with a cell structure.

• Compute the fundamental group of $(A, a)$.

• Produce a 2-sheeted covering map $\pi : (B, b) \to (A, a)$, with $B$ path-connected.

• Compute the fundamental group of $(B, b)$. Compute its image $\text{im}(\pi_*) \subset \pi_1(A, a)$.

• Produce a 2-sheeted covering map $\tau : (C, c) \to (A, a)$, not isomorphic to $\pi$, with $C$ path-connected.

You have to justify that $\pi$ and $\tau$ are indeed covering.