Final exam WISB326, June 26, 2023, 09:00-12:00

Exam problems

- Let k be an algebraically closed field of characteristic 0.

(1) (13 points) Let $X = V(x^2y^3 + x^2y^2wz^3, y^4 - y^2w^2z^6) \subseteq \mathbb{A}^4(k)$. Compute the irreducible components of X, and for each irreducible component Y of X compute the dimension of Y.

(2) (8 points) Let $I \subseteq k[x_0, \dots, x_n]$ be a homogeneous ideal. Show that the radical of I is a homogeneous ideal.

(3) For every $a = (a_{0,0}, a_{0,1}, a_{0,2}, a_{1,1}, a_{1,2}, a_{2,2}) \in k^6$, let $f_a = \sum_{0 \le i \le j \le 2} a_{i,j}x_ix_j$. Let X be the set of points in $\mathbb{A}^6(k)$ such that f_a defines a projective plane curve such that $(1 : 1 : 0) \in V(f_a)$.

(a) (3 points) Show that $X \subseteq \mathbb{A}^6(k) \setminus \{(0, 0, 0, 0, 0, 0)\}$ is a closed subset.

(b) (3 points) Show that $X \subseteq \mathbb{A}^6(k)$ is not an algebraic set.

(4) (5 points) Let $f \in k[x, y, z]$ be an irreducible homogeneous polynomial of degree 2. Show that all the points in $V(f) \subseteq \mathbb{P}^2(k)$ are nonsingular for f.

(5) Consider the projective plane curves $X = V(f)$ and $Y = V(g)$ given by the polynomials

$$f = x_0^5 - x_0x_1^4 + 2x_1(x_0^4 - x_2^4) \qquad g = x_0^5 - x_0x_1^3x_2 + 2x_1(x_0^4 - x_2^4).$$

(a) (8 points) Show that $I((0 : 0 : 1), f \cap g) = 16$.

(b) (12 points) Compute all the points in the intersection $X \cap Y$, and for each point $P \in X \cap Y$ compute the intersection multiplicity $I(P, f \cap g)$.

(c) (3 points) Determine multiplicity, tangent lines and multiplicity of the tangent lines at the point $(0 : 1 : 0)$ for the projective plane curve g.

(6) (6 points) Let $f_0, \dots, f_s \in k[x_0, \dots, x_r]$ be homogeneous polynomials of degree d. Let $U = \mathbb{P}^r(k) \setminus V(f_0, \dots, f_s)$. Show that $f = (f_0, \dots, f_s) : U \to \mathbb{P}^s(k)$ is a morphism.

(7) Let

$$\varphi : \mathbb{A}^1(k) \to \mathbb{A}^3(k), \quad t \to (t^3, t^4, t^7 - 1).$$

Let $C = \varphi(\mathbb{A}^1(k))$.

(a) (4 points) Show that C is irreducible.

(b) (4 points) Show that C is a curve.

(c) (7 points) Show that φ is birational and find an explicit inverse map ψ.

(d) (4 points) Show that ψ is not a morphism.