Measure and Integration: Retake Final Exam 2022-23
You are (only) allowed to use the textbook of the course

(1) Let $(X,\mathcal{A},\mu)$ be a measure space and $\mathcal{G} = \{A_1, A_2, \cdots\}$ be a countable partition of X with $A_k \in \mathcal{A}$ for all $k \in \mathbb{N}$. Define a function $u : X \to \mathbb{R}$ by $u(x) = \sum_{k=1}^{\infty} k \cdot \mathbb{I}_{A_k}$.

(a) Prove that u is $\mathcal{A}/\mathcal{B}(\overline{\mathbb{R}})$-measurable. (0.5 pt)

(b) Prove that $\sigma(u) = \sigma(\mathcal{G})$, where $\sigma(u)$ is the smallest σ-algebra making u $\mathcal{A}/\mathcal{B}(\overline{\mathbb{R}})$-measurable. (1.5 pts)

(c) Suppose that $0 < \mu(A_n) < \infty$ for all $n \in N$. Define ν on $\mathcal{A}$ by

$$\nu(B) = \sum_{n=1}^{\infty} 3^{-n} \cdot \frac{\mu(B \cap A_n)}{\mu(A_n)}.$$

Prove that ν is a **finite** measure on $(X,\mathcal{A})$. (1 pt)

(2) Consider the measure space $([0,1],\mathcal{B}([0,1]),\lambda)$, where $\mathcal{B}([0,1])$ is the restriction of the Borel σ-algebra on $[0,1]$, and λ Lebesgue measure on $[0,1]$. Let $p \in (1,\infty)$ and let q be the conjugate of p, i.e. $\frac{1}{p} + \frac{1}{q}$. Assume $f \in \mathcal{L}^p(\lambda)$.

(a) Prove that

$$n^{1/q} \int_{[0,1/n]} |f|\, d\lambda \le \left(\int_{[0,1/n]} |f|^p\, d\lambda \right)^{1/p}.$$

(1.5 pts)

(b) Prove that $\lim_{n\to\infty} n^{1/q} \int_{[0,1/n]} |f|\, d\lambda = 0$. (1 pt)

(3) Let $X = Y = \mathbb{Z}_+ = \{0,1,2,3,\cdots\}$. Let $\mathcal{A}$ be the collection of all subsets of $\mathbb{Z}_+$ and $\mu_1 = \mu_2$ be counting measure on $\mathbb{Z}_+$. Let $u : X \times Y \to \mathbb{R}$ be defined by

$$u(n,m) = \begin{cases} 1 + 2^{-n} & \text{if } n = m, \\ -1 - 2^{-n} & \text{if } n = m+1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Prove that $\int_X \int_Y u(n,m)\, d\mu_2(m)\, d\mu_1(n) = 2$. (1 pt)

(b) Prove that $\int_Y \int_X u(n,m)\, d\mu_1(n)\, d\mu_2(m) = 1$. (1 pt)

(c) Explain why parts (a) and (b) do not contradict Fubini's Theorem. (0.5 pt)

(4) Let $(X,\mathcal{A},\mu)$ be a measure space. Assume $u \in \mathcal{M}^+$ satisfies $\int u\, d\mu < \infty$ and let $\epsilon > 0$.

(a) Show that there exists a simple function $f \in \mathcal{E}^+$ such that $0 \le f \le u$ and

$$\int u\, d\mu - \int f\, d\mu < \epsilon/2.$$

(0.5 pt)

(b) Let $M = \max\{f(x) : x \in X\}$, where f is the simple function obtained in part (a). Prove that for any $B \in \mathcal{A}$, one has

$$\int_B u\, d\mu \le \frac{\epsilon}{2} + M\mu(B).$$

(1 pt)

(c) Prove that there exists $\delta > 0$ such that $\int_B u\, d\mu < \epsilon$ for any $B \in \mathcal{A}$ with $\mu(B) < \delta$. (0.5 pt)