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Measure and Integration: Mid-Term, 2022-23

(1) Let X be a set and $\mathcal{B}$ a collection of subsets of X satisfying the following two properties: (i) $X \in \mathcal{B}$, (ii) if $A, B \in \mathcal{B}$, then $A \setminus B = A \cap B^c \in \mathcal{B}$.

(a) Prove that if $A \in \mathcal{B}$, then $A^c \in \mathcal{B}$. (0.5 pt)

(b) Prove that if $A, B \in \mathcal{B}$, then $A \cup B \in \mathcal{B}$. (0.5 pt)

(c) Suppose that $\mathcal{B}$ satisfies the additional property:
(iii) for every **decreasing** sequence $(A_n)_{n\in\mathbb{N}}$ of sets in $\mathcal{B}$, one has $\bigcap_{n\in\mathbb{N}} A_n \in \mathcal{B}$.

Prove that $\mathcal{B}$ is a σ-algebra. (2 pts)

(2) Consider the measure space $([0,1), \mathcal{B}([0,1)), \lambda)$, where $\mathcal{B}([0,1))$ is the Borel σ-algebra restricted to $[0,1)$ and λ is the restriction of Lebesgue measure on $[0,1)$. Define a map $T : [0,1) \to [0,1)$ by

$$T(x) = \sum_{n=0}^{\infty} \left(2^{n+1}x - 1\right) \cdot \mathbb{I}_{[2^{-(n+1)}, 2^{-n})}(x),$$

where $\mathbb{I}_A$ denotes the indicator function of the set A.

(a) Show that T is $\mathcal{B}([0,1))/\mathcal{B}([0,1))$ measurable. (2 pt)

(b) Determine the image measure $T(\lambda) = \lambda \circ T^{-1}$ and prove that $T(\lambda) = \lambda$. (2 pts)

(3) Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra and λ is Lebesgue measure. Let $B \in \mathcal{B}(\mathbb{R})$ be such that $0 < \lambda(B) < \infty$, and define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \lambda\Big(B \cap (-\infty, x]\Big).$$

(a) Prove that f is an increasing and uniformly continuous function. (1 pt)

(b) Prove that $\lim_{x\to+\infty} f(x) = \lambda(B)$ and $\lim_{x\to-\infty} f(x) = 0$. (1 pt)

(c) Prove that for any $\beta \in (0, \lambda(B))$ there exists a Borel measurable subset C_β of B such that $\lambda(C_\beta) = \beta$. (Hint: use the Intermediate Value Theorem) (1 pt)