Final exam, Numerical Analysis (WISB251)

Tuesday, 11 April 2023, 17:00-20:00, BBG 023, 061

- Write your name on each page you turn in, and additionally, on the first page, write your student number and the *total number of pages submitted.*
- You may use one A4 sheet with notes while working the problems.
- For each question, motivate your answer. You may make use of results from previous subproblems, even if you have been unable to prove them.
- The maximum number of points per subproblem are given between square brackets. Your grade is the total earned points divided by 4. The final exam weighs 50% in your grade for the course.

Problem 1. [*Nonlinear systems of algebraic equations*]

Consider the following system of nonlinear equations for x and y. Write $r = (x, y)^T$.

$$f(r) = \begin{pmatrix} x + \frac{1}{2}y - \frac{\pi}{2} \\ y - \frac{1}{2}\sin(x + \frac{1}{2}y) \end{pmatrix} = 0$$

Suppose we attempt to solve this system using the fixed point iteration

$$r_{k+1} = r_k - \alpha f(r_k).$$

(a) [2pts] Find (by hand) a solution $r^* = (x^*, y^*)^T$, $f(r^*) = 0$ of the nonlinear system.

(b) [4pts] What is the Jacobian matrix $f'(r^*) = Df(r^*)$ at r^*? What are its eigenvalues?

(c) [4pts] For what range of values α does the fixed-point iteration converge to r^*?

Problem 2. [*Numerical integration*]

We wish to approximate the definite integral

$$I = \int_{-1}^{1} f(x)\, dx,$$

using the values $f(c)$ and $f(-c)$ for $0 < c \leq 1$.

(a) [3pt] Construct the interpolating polynomial $p(x)$ through the points $(c, f(c))$ and $(-c, f(-c))$.

(b) [2pt] Show that the associated quadrature formula is given by

$$\bar{I} = \int_{-1}^{1} p(x)\, dx = f(c) + f(-c).$$

(c) [3pt] Derive an expression for the error $E = I - \bar{I}$ for the case $f(x)$ is a polynomial of degree n.

(d) [2pt] The formula is exact for polynomials up to a certain degree n. For what carefully chosen value of c can you maximize this degree n?

Problem 3. [*Numerical integration of ODEs*]

Consider the following numerical method for solving an initial value problem $y'(t) = f(y(t))$, $y(0) = y_0$, $y(t) \in \mathbf{R}^d$, $f : \mathbf{R}^d \to \mathbf{R}^d$, $t \in [0, T]$:

$$y_{n+1} = y_n + hf\left((1-\theta)y_n + \theta y_{n+1}\right),$$

where $y_n \approx y(t_n)$, $n = 0, \ldots, N$, $t_n = nh$, $h = T/N$.

(a) [5pts] Determine the truncation error for this method in the form

$$\text{trunc. error} = Ch^q + \mathcal{O}(h^{q+1})$$

(i.e. determine q and and expression for C). What choice of θ gives the best accuracy? (*Hint:* Define $\bar{y}(t) = (1-\theta)y(t) + \theta y(t+h)$, and derive the Taylor expansion of $\bar{y}(t)$ about $y(t)$. Then determine the Taylor expansion of $f(\bar{y}(t))$ about $y(t)$.)

(b) [3pts] Compute the stability function $R(z)$ such that $y_{n+1} = R(h\lambda)y_n$ when the method is applied to the test problem $y'(t) = \lambda y(t)$ for λ a complex number.

(c) [2pts] Sketch the stability regions $S = \{z \in \mathbb{C}; |R(z)| < 1\}$ for $\theta = 0$, $\theta = 1$, and $\theta = 1/2$.

Problem 4. [*Numerical differentiation formula*]

In some applications it is necessary to evaluate a numerical difference formula at the midpoint between two nodes.

(a) [2pts] Write the Newton divided difference polynomial $p_1(x)$ for the points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ with $x_1 = x_0 + h$, and give an expression for the error $e(x) = f(x) - p_1(x)$.

(b) [3pts] Show that the approximation of the derivative $f'(x) = p_1'(x)$ is given by the difference formula

$$f'(x) \approx \frac{f(x_0 + h) - f(x_0)}{h},$$

(independent of x) and derive an upper bound on the error $e'(x) = f'(x) - p_1'(x)$ for $x \in [x_0, x_1]$.

(c) [2pts] By directly expanding $f(x_0 + h)$ in a Taylor series about x_0, derive an error bound for the above difference formula at x_0.

(d) [3pts] Instead, expand both $f(x_0 + h)$ and $f(x_0)$ in a Taylor series about the midpoint $\widehat{x} = x_0 + \frac{h}{2}$, and derive an error bound for the difference formula at $\widehat{x}$. How does the error of the approximation at the midpoint compare with your earlier bounds?