Solutions - Introduction to Financial Mathematics

1. Let $\{W(t) : t \geq 0\}$ be a Brownian motion, we define process $\{X(t) : t \geq 0\}$

$$X(t) = \frac{1}{\sqrt{3}} W(3t).$$

a. Prove that $\{X(t) : t \geq 0\}$ is a Brownian motion. (10 pts.)

b. Let $Y(t) = X^2(t) - 2\sqrt{c}t$ for some non-negative constant c and for all $t \geq 0$. For which value of c is the process $\{Y(t) : t \geq 0\}$ a martingale with respect to the filtration $\{\mathcal{F}(t) : t \geq 0\}$, with $\mathcal{F}(t) = \sigma(X(s) : s \leq t)$? (10 pts.)

Proof **1(a)**: We check that the process $(X(t) : t \geq 0)$ satisfies all the properties of a Brownian motion. We have

(i) $X(0) = \frac{1}{\sqrt{3}} W(0) = 0$.

(ii) Since $(W(t) : t \geq 0)$ has continuous paths and the function $t \to 3t$ is continuous and so is multiplication by $1/\sqrt{3}$, we see that the process $(X(t) : t \geq 0)$ also has continuous paths.

(iii) If $0 \leq u < v < s < t$, then clearly $0 \leq 3u < 3v < 3s < 3t$. Hence $W(3v) - W(3u)$ and $W(3t) - W(3s)$ are independent, implying that $X(v) - X(u)$ and $X(t) - X(s)$ are independent. Therefore the process $(X(t) : t \geq 0)$ has independent increments.

(iv) if $s < t$, then $W(3t) - W(3s)$ is normally distributed with mean zero and variance $3t - 3s = 3(t - s)$. hence $X(t) - X(s)$ is also normally distributed with mean

$$\mathbb{E}[X(t) - X(s)] = \frac{1}{\sqrt{3}} \mathbb{E}[W(3t) - W(3s)] = 0,$$

and variance

$$\mathbb{V}ar(X(t) - X(s)) = \frac{1}{3} \mathbb{V}ar(W(3t) - W(3s)) = (t - s).$$

Therefore, $(X(t) : t \geq 0)$ is a Brownian motion.

Proof **1(b)** The underlying filtration is given by $\mathcal{F}(t) = \sigma(X(s) : s \leq t)$. Now let $s < t$, and note that $Y(t) - Y(s)$ is independent of $\mathcal{F}(s)$ while $Y(s)$ is $\mathcal{F}(s)$-measurable. Hence,

$$\begin{aligned}
\mathbb{E}[Y(t)|\mathcal{F}(s)] &= \mathbb{E}[(Y(t) - Y(s)) + Y(s)|\mathcal{F}(s)] \\
&= \mathbb{E}[Y(t) - Y(s))] + Y(s) \\
&= \mathbb{E}[X^2(t) - X^2(s)] - 2\sqrt{c}(t - s) + Y(s) \\
&= \mathbb{E}[\frac{1}{3} W^2(3t) - \frac{1}{3} W^2(3s)] - 2\sqrt{c}(t - s) + Y(s) \\
&= (1/3)3t - 3s - 2\sqrt{c}(t - s) + Y(s) \\
&= (1 - 2\sqrt{c})(t - s) + Y(s).
\end{aligned}$$

Hence, for the process $(Y(t) : t \geq 0)$ to be a martingale, we must have $c = 1/4$.

2. Let $\{W(t) : 0 \le t \le T\}$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\{\mathcal{F}(t) : 0 \le t \le T\}$ be its natural filtration, and assume $\mathcal{F} = \mathcal{F}(T)$.

Consider a stock with price process $\{S(t) : 0 \le t \le T\}$ with

$$S(t) = S(0)\exp\left\{\int_0^t \mathrm{e}^{-u}\mathrm{d}W(u) + \int_0^t (1 - \frac{1}{2}\mathrm{e}^{-2u})\mathrm{d}u\right\}$$

(a) Let

$$X(t) = \int_0^t \mathrm{e}^{-u}\mathrm{d}W(u) + \int_0^t (1 - \frac{1}{2}\mathrm{e}^{-2u})\mathrm{d}u$$

Determine the distribution of $X(t)$.

(b) Prove the $\{S(t) : t \ge 0\}$ is an Itô process.

Proof **2(a)**: Let $Y(t) = \int_0^t \mathrm{e}^{-u}\mathrm{d}W(u)$. Since $Y(t)$ is the Itô integral of a deterministic process, by Theorem 4.4.9 $Y(t)$ is normally distributed with $\mathbb{E}[Y(t)] = 0$ and $\mathbb{V}ar[Y(t)] = \int_0^t \mathrm{e}^{-2u}\mathrm{d}u = \frac{1}{2}(1 - \mathrm{e}^{-2t})$. Since $X(t) = Y(t) + \int_0^t (1 - \frac{1}{2}e^{-u})\mathrm{d}u = Y(t) + t + \frac{1}{4}(\mathrm{e}^{-2t} - 1)$, we see that $X(t)$ is normally distributed with mean $\mathbb{E}[X(t)] = t + \frac{1}{4}(\mathrm{e}^{-2t} - 1)$ and variance $\mathbb{V}ar[X(t)] = \mathbb{V}ar[Y(t)] = \frac{1}{2}(1 - \mathrm{e}^{2t})$.

Proof **2(b)**: With $X(t) = \int_0^t \mathrm{e}^{-u}\mathrm{d}W(u) + \int_0^t (1 - \frac{1}{2}\mathrm{e}^{-2u})\mathrm{d}u$ we have $\mathrm{d}X(t) = \mathrm{e}^{-t}\mathrm{d}W(t) + (1 - \frac{1}{2}\mathrm{e}^{-2t})\mathrm{d}t$ and $\mathrm{d}X(t)\mathrm{d}X(t) = \mathrm{e}^{-2t}\mathrm{d}t$. Note that $S(t) = S(0)\mathrm{e}^{X(t)}$, so let $f(x) = S(0)\mathrm{e}^x$, then $f_x(x) = f_{xx}(x) = f(x)$. By the Itô-Doeblin formula, we have,

$$\begin{aligned}
\mathrm{d}S(t) &= \mathrm{d}f(X(t)) = S(t)\mathrm{d}X(t) + \frac{1}{2}S(t)\mathrm{d}X(t)\mathrm{d}X(t) \\
&= S(t)\left(\mathrm{e}^{-t}\mathrm{d}W(t) + (1 - \frac{1}{2}\mathrm{e}^{-2t})\mathrm{d}t\right) + \frac{1}{2}S(t)\mathrm{e}^{-2t}\mathrm{d}t \\
&= S(t)\mathrm{d}t + S(t)\mathrm{e}^{-t}\mathrm{d}W(t).
\end{aligned}$$

This shows that $S(t) = S(0) + \int_0^t S(u)\mathrm{d}u + \int_0^t S(u)\mathrm{e}^{-u}\mathrm{d}W(u)$. Hence, $\{S(t) : t \ge 0\}$ is an Itô process.

3. Let $\{W(t) : 0 \le t \le T\}$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\{\mathcal{F}(t) : 0 \le t \le T\}$ be its natural filtration, and assume $\mathcal{F} = \mathcal{F}(T)$. Consider a stock with price process $\{S(t) : 0 \le t \le T\}$ with $S(t) = t^4 + 4W(t)$.

(a) Construct a measure $\tilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ (i.e., $\tilde{\mathbb{P}}(A) = 0$ if and only if $\mathbb{P}(A) = 0,\ A \in \mathcal{F}$), such that the price process $\{S(t) : 0 \le t \le T\}$ is a martingale under $\tilde{\mathbb{P}}$ and with respect to the filtration $\{\mathcal{F}(t) : 0 \le t \le T\}$. (10 pts.)

(b) Consider a European call option on this stock with expiration date T and strike price K. Find an expression for

$$C(0) = \tilde{\mathbb{E}}[\mathrm{e}^{-rT}(S(T) - K)^+],$$

the price of this option at time 0. (with interest rate r). (10 pts.)

Proof **3(a)**: Define $\theta(t) = t^3$, then

$$\frac{S(t)}{4} = \int_0^t \theta(u)\mathrm{d}u + W(t).$$

Consider the random variable Z ($= Z(T)$, as given in Girsanov's Theorem), defined by

$$Z = \exp\left(-\int_0^T \theta(u)\mathrm{d}W(u) - \frac{1}{2}\int_0^T \theta^2(u)\mathrm{d}u\right) = \exp\left(-\int_0^T u^3\mathrm{d}W(u) - \frac{T^7}{14}\right).$$

Note that $\int_0^t \theta(u)\mathrm{d}W(u)$, $\int_0^t \theta^2(u)\mathrm{d}u$ and θ are continuous functions on the compact interval $[0,T]$, hence they are all bounded and the same holds for Z. This implies that $\mathbb{E}\left[\int_0^T \theta^2(u)Z^2(u)\mathrm{d}u\right] < \infty$. Define the measure $\tilde{\mathbb{P}}$ on $\mathcal{F}$ by $\tilde{\mathbb{P}}(A) = \int_A Z$. By Girsanov's Theorem, the process $\{\tilde{W}(t) = \frac{S(t)}{4} : 0 \le t \le T\}$ is a Brownian motion under $\tilde{\mathbb{P}}$ and hence it is a martingale under $\tilde{\mathbb{P}}$. Since multiplying a martingale with a constant remains a martingale, we see that $\{S(t) : 0 \le t \le T\}$ is a martingale under $\tilde{\mathbb{P}}$.

Proof **3(b)**: By part (a), we see that under the measure $\tilde{\mathbb{P}}$, the random variable $S(T)/4$ is $N(0,T)$ distributed, hence $S(T)$ is $N(0,16T)$, and

$$\begin{aligned}
C(0) &= \tilde{\mathbb{E}}\left[\mathrm{e}^{-rT}(S(T)-K)^+\right] \\
&= \mathrm{e}^{-rT}\int_{-\infty}^{\infty}(x-K)^+\frac{1}{\sqrt{32\pi T}}\mathrm{e}^{-\frac{x^2}{32T}}\mathrm{d}x \\
&= \mathrm{e}^{-rT}\int_{K}^{\infty}(x-K)\frac{1}{\sqrt{32\pi T}}\mathrm{e}^{-\frac{x^2}{32T}}\mathrm{d}x \\
&= \mathrm{e}^{-rT}\int_{K/4\sqrt{T}}^{\infty}(4\sqrt{T}y-K)\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{y^2}{2}}\mathrm{d}y \\
&= \mathrm{e}^{-rT}\int_{K/4\sqrt{T}}^{\infty}4\sqrt{T}y\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{y^2}{2}}\mathrm{d}y - Ke^{-rT}\int_{K/4\sqrt{T}}^{\infty}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{y^2}{2}}\mathrm{d}y \\
&= \mathrm{e}^{-rT-K^2/32T}4\sqrt{\frac{T}{2\pi}} - \mathrm{e}^{-rT}K\left(1-N(K/4\sqrt{T})\right)
\end{aligned}$$

where $N(y)$ is the standard normal distribution function.

4. Given a Radon-Nikodym derivative Z, and the associated Radon-Nikodym process $\{Z(t) : t \ge 0\}$, defined by $Z(t) = \mathbb{E}[Z|\mathcal{F}(t)]$, where $\{\mathcal{F}(t) : t \ge 0\}$ is a given filtration. We then have the change of probability measure, $\mathrm{d}\tilde{\mathbb{P}} = Z\mathrm{d}\mathbb{P}$, with the expectation under the $\tilde{\mathbb{P}}$-measure, i.e., $\tilde{\mathbb{E}}[Y] = \mathbb{E}[ZY]$

a. Let Y be a random variable which is $\mathcal{F}(t)$-measurable. Prove that

$$\mathbb{E}[YZ] = \tilde{\mathbb{E}}[Y] = \mathbb{E}[YZ(t)]. \quad \text{(10 pts.)}$$

b. Suppose Y is $\mathcal{F}(t)$-measurable, then prove (using partial averaging) that, for $s < t$

$$\tilde{\mathbb{E}}[Y|\mathcal{F}(s)] = \frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}(s)]. \quad \text{(10 pts.)}$$

Proof **4(a)**: We look for the proof of Lemma 5.2.1 from the book. Recall: $\tilde{\mathbb{E}}[Y] = \mathbb{E}[ZY]$, Y is a r.v. so Y is $\mathcal{F}(t)$-measurable.

Let $\{\mathcal{F}(t) : t \geq 0\}$ be a given filtration (for which we have defined the Radon-Nikodym process).

We consider the RHS

$$\mathbb{E}[YZ(t)] = \mathbb{E}[Y\mathbb{E}[Z|\mathcal{F}(t)] = \mathbb{E}[\mathbb{E}[YZ|\mathcal{F}(t)]] = \mathbb{E}[YZ] = \tilde{\mathbb{E}}[Y]$$

using the $\mathcal{F}(t)$ measurability of Y.

Proof **4(b)**: Here we look for the proof of book's Lemma 5.2.2:
Recall: $\tilde{\mathbb{E}}[Y] = \mathbb{E}[ZY]$, Y is a r.v., so Y is $\mathcal{F}(t)$-measurable. To prove the result, it is enough to show that the RHS is the conditional expectation of Y given $\mathcal{F}(s)$ under the measure $\tilde{\mathbb{P}}$.

So, we need to verify the two defining conditions of conditional expectations.

(i) Clearly the RHS is $\mathcal{F}(s)$-measurable. $Z(s)^{-1}$ is $\mathcal{F}(s)$-measurable; the same holds for the second term. Hence, the product is $\mathcal{F}(s)$-measurable.

(ii) Now, let $A \in \mathcal{F}(s)$, we want to show

$$\begin{aligned}
\int_A \frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}(s)]\mathrm{d}\tilde{\mathbb{P}} &= \int_A Y\mathrm{d}\tilde{\mathbb{P}} = \tilde{\mathbb{E}}[\mathbb{1}_A Y] \\
\int_A \frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}(s)]\mathrm{d}\tilde{\mathbb{P}} &= \tilde{\mathbb{E}}[\mathbb{1}_A \frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}(s)]] \\
&= \tilde{\mathbb{E}}[\mathbb{E}[\mathbb{1}_A \frac{1}{Z(s)}\, YZ(t)|\mathcal{F}(s)]] \text{ use Lemma 5.2.1} \\
&= \mathbb{E}[Z(s)\mathbb{E}[\mathbb{1}_A \frac{1}{Z(s)} YZ(t)|\mathcal{F}(s)]] \\
&= \mathbb{E}[\mathbb{E}[\mathbb{1}_A YZ(t)|\mathcal{F}(s)]] = \mathbb{E}[\mathbb{1}_A YZ(t)] \\
\text{(using again Lemma 5.2.1)} &= \tilde{\mathbb{E}}[1_A Y] = \int_A Y\mathrm{d}\tilde{\mathbb{P}} \\
\text{So, } \frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}(s)] &= \tilde{\mathbb{E}}[Y|\mathcal{F}(s)]
\end{aligned}$$

5. Let $\{(W_1(t), W_2(t)) : t \geq 0\}$ be a 2-dimensional Brownian motion, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider two price processes $\{S_1(t) : t \geq 0\}$ and $\{S_2(t) : t \geq 0\}$ with corresponding SDE given by:

$$\begin{aligned}
\mathrm{d}S_1(t) &= 2\alpha S_1(t)\mathrm{d}W_1(t) + 3\beta S_1(t)\mathrm{d}W_2(t) \\
\mathrm{d}S_2(t) &= \frac{\gamma}{2} S_2(t)\mathrm{d}t + \frac{\sigma}{4} S_2(t)\mathrm{d}W_1(t),
\end{aligned}$$

where $\alpha, \beta, \gamma, \sigma$ are positive constants.

a. Show that $\{S_1(t)S_2(t) : t \geq 0\}$ is a 2-dimensional Itô-process. (10 pts.)

b. Consider a finite expiration date T, and suppose the interest rate is constant, $R(t) = r$ for all $t > 0$. Show that the market price equations have a unique solution, and determine the risk-neutral probability measure $\tilde{\mathbb{P}}$ for the process $\{(S_1(t), S_2(t) : 0 \leq t \leq T\}$. (10 pts.)

Proof 5(a): We apply Itô's product rule, as follows,

$$\mathrm{d}(S_1(t)S_2(t)) = S_1(t)\mathrm{d}S_2(t) + S_2(t)\mathrm{d}S_1(t) + \mathrm{d}S_1(t)\mathrm{d}S_2(t).$$

Using $\mathrm{d}S_1(t) = 2\alpha S_1(t)\mathrm{d}W_1(t) + 3\beta S_1(t)\mathrm{d}W_2(t)$, $\mathrm{d}S_2(t) = \gamma/2 S_2(t)\mathrm{d}t + \sigma/4 S_2(t)\mathrm{d}W_1(t)$, and simplifying, we get

$$\begin{aligned}\mathrm{d}(S_1(t)S_2(t)) &= (\gamma/2 + (\alpha\sigma)/2)S_1(t)S_2(t)\mathrm{d}t \\ &\quad + (\sigma/4 + 2\alpha)S_1(t)S_2(t)\mathrm{d}W_1(t) + 3\beta S_1(t)S_2(t)\mathrm{d}W_2(t).\end{aligned}$$

Equivalently,

$$\begin{aligned}S_1(t)S_2(t) &= S_1(0)S_2(0) + \int_0^t (\gamma/2 + (\alpha\sigma)/2)S_1(u)S_2(u)\mathrm{d}u \\ &\quad + \int_0^t (\sigma/4 + 2\alpha)S_1(u)S_2(u)\mathrm{d}W_1(u) + \int_0^t 3\beta S_1(u)S_2(u)\mathrm{d}W_2(u).\end{aligned}$$

Hence, $\{S_1(t)S_2(t) : t \geq 0\}$ is a 2-dimensional Itô process.

Proof 5(b): Using the notation of the book, we have

$$\alpha_1 = 0, \sigma_{11} = 2\alpha, \sigma_{12} = 3\beta, \alpha_2 = \gamma/2, \sigma_{21} = \sigma/4, \sigma_{22} = 0.$$

The market price equations in this case form the system,

$$\begin{aligned}-r &= 2\alpha\theta_1(t) + 3\beta\theta_2(t) \\ \gamma/2 - r &= \sigma/4\theta_1(t)\end{aligned}$$

Solving for $\theta_1(t), \theta_2(t)$, we get

$$\begin{aligned}\theta_1(t) &= \frac{\gamma/2 - r}{\sigma/4} \\ \theta_2(t) &= -\frac{\sigma/4r + 2\alpha(\gamma/2 - r)}{(\sigma/4)3\beta}\end{aligned}$$

Setting

$$\begin{aligned}Z &= \exp\left\{-\int_0^T (\theta_1(t)\mathrm{d}W_1(t) + \theta_2(t)\mathrm{d}W_2(t)) - \frac{1}{2}\int_0^T \left(\theta_1^2(t) + \theta_2^2(t)\right)\mathrm{d}t\right\} \\ &= \exp\left\{-\frac{\gamma/2 - r}{\sigma/4}W_1(T) + \frac{\sigma/4r + 2\alpha(\gamma/2 - r)}{(\sigma/4)3\beta}W_2(T)\right. \\ &\quad \left.-\frac{1}{2}\left(\frac{(\gamma/2 - r)^2}{(\sigma/4)^2} + \frac{(\sigma/4r + 2\alpha(\gamma/2 - r))^2}{(\sigma/4)^2 3\beta^2}\right)T\right\}\end{aligned}$$

(2)

The risk-neutral measure is given by $\tilde{\mathbb{P}}(A) = \int_A Z \mathrm{d}\mathbb{P}$. To check this, we set

$$\tilde{W}_1(t) = \frac{\gamma/2 - r}{\sigma/4} t + W_1(t)$$

and

$$\tilde{W}_2(t) = -\frac{\sigma/4r + 2\alpha(\gamma/2 - r)}{(\sigma/4)3\beta} t + W_2(t).$$

By the 2-dimensional Girsanov Theorem, the process $\{(\tilde{W}_1(t), \tilde{W}_2(t)) : 0 \le t \le T\}$ is a 2-dimensional Brownian motion under $\tilde{\mathbb{P}}$.
Rewriting $\mathrm{e}^{-rt}S_1(t), \mathrm{e}^{-rt}S_2(t)$ in terms of $\tilde{W}_1(t), \tilde{W}_2(t)$, we get after applying the Itô product rule

$$\begin{aligned} \mathrm{d}(\mathrm{e}^{-rt}S_1(t)) &= \mathrm{e}^{-rt}S_1(t)(2\alpha \mathrm{d}\tilde{W}_1(t) + 3\beta \mathrm{d}\tilde{W}_2(t)) \\ \mathrm{d}(\mathrm{e}^{-rt}S_2(t)) &= \mathrm{e}^{-rt}S_2(t)\sigma/4 \mathrm{d}\tilde{W}_1(t), \end{aligned}$$

which shows that the discounted price processes are Itô integrals and hence martingales under $\tilde{\mathbb{P}}$.

6. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\{W(t) = (W_1(t), W_2(t)) : t \ge 0\}$ a 2-dim. Brownian Motion with filtration $\{\mathcal{F}(t) : 0 \le t \le T\}$ (T is a fixed time). Let $\{\theta(t) = (\theta_1(t), \theta_2(t)) : 0 \le t \le T\}$ be an adapted process. Define,

$$Z(t) = \exp\left\{-\int_0^t \theta_1(u)\mathrm{d}W_1(u) - \int_0^t \theta_2(u)\mathrm{d}W_2(u) - \frac{1}{2}\int_0^t (\theta_1^2(u) + \theta_2^2(u))\mathrm{d}u\right\}$$

and $\tilde{W}_i(t) = W_i(t) + \int_0^t \theta_i(u)\mathrm{d}u), \ \ i = 1, 2$.

Moreover, assume

$$\mathbb{E}\left[\int_0^t (\theta_1^2(u) + \theta_2^2(u))Z(u)\mathrm{d}u\right] < \infty,$$

set $Z = Z(T)$ and define $\tilde{\mathbb{P}}(A) = \int_A Z \mathrm{d}\mathbb{P}$.

Prove that $\{\tilde{W}(t) : 0 \le t \le T\}$ is a 2-dim. Brownian Motion under $\tilde{\mathbb{P}}$. (10 pts.)

Proof **6**: The proof follows the same steps as in the one-dimensional case. We use here the 2-dimensional Lévy characterization. It is easily checked that (student has to show this)
(i) $\tilde{W}_1(0) = \tilde{W}_2(0) = 0$;
(ii) $\{\tilde{W}_i(t) : 0 \le t \le T\}$ has continuous paths, $i = 1, 2$;
(iii) $[\tilde{W}_i, \tilde{W}_i](t) = [W_i, W_i](t) = t$;
(iv) $[\tilde{W}_i, \tilde{W}_j](t) = 0, \ i \ne j$.
The only thing left to prove is that $\{\tilde{W}_1(t) : 0 \le t \le T\}$ and $\{\tilde{W}_2(t) : 0 \le t \le T\}$ are martingales under $\tilde{\mathbb{P}}$.

Claim: $\{\tilde{W}_i(t)Z(t) : 0 \le t \le T\}$ is a martingale under $\mathbb{P}$.

Proof of claim: We have,

$$\begin{aligned}
\mathrm{d}Z(t) &= -Z(t)\theta_1(t)\mathrm{d}W_1(t) - Z(t)\theta_2(t)\mathrm{d}W_2(t) \\
\mathrm{d}\tilde{W}_i(t) &= \mathrm{d}W_i(t) + \theta_i(t)\mathrm{d}t,\ i = 1,2
\end{aligned}$$

We apply the Itô-product rule, to get $\mathrm{d}(\tilde{W}_1(t)Z(t)) =$

$$\begin{aligned}
&= Z(t)\mathrm{d}\tilde{W}_1(t) + \tilde{W}_1(t)\mathrm{d}Z(t) + \mathrm{d}Z(t)\mathrm{d}\tilde{W}_1(t) \\
&= Z(t)(\mathrm{d}W_1(t) + \theta_1(t)\mathrm{d}t) \\
&\quad + \tilde{W}_1(t)(-Z(t)\theta_1(t)\mathrm{d}W_1(t) - Z(t)\theta_2(t)\mathrm{d}W_2(t)) \\
&\quad (-Z(t)\theta_1(t)\mathrm{d}W_1(t) - Z(t)\theta_2(t)\mathrm{d}W_2(t))(\mathrm{d}W_1(t) + \theta_1(t)\mathrm{d}t) \\
&= Z(t)\mathrm{d}W_1(t) + Z(t)\theta_1(t)\mathrm{d}t - Z(t)\tilde{W}_1(t)\theta_1(t)\mathrm{d}W_1(t) \\
&\quad - \tilde{W}_1(t)Z(t)\theta_2(t)\mathrm{d}W_2(t) - Z(t)\theta_1(t)\mathrm{d}t \\
&= Z(t)\left(1 - \theta_1(t)\tilde{W}_1(t)\right)\mathrm{d}W_1(t) - \tilde{W}_1(t)Z(t)\theta_2(t)\mathrm{d}W_2(t)
\end{aligned}$$

$\Rightarrow$ This is a sum of two Itô-integrals. Hence $\{\tilde{W}_1(t)Z(t) : 0 \leq t \leq T\}$ is a martingale under $\mathbb{P}$, and similarly $\{\tilde{W}_2(t)Z(t) : 0 \leq t \leq T\}$ is a martingale under $\mathbb{P}$.

We now use Lemma 5.2.2. to get for $s < t$:

$$\begin{aligned}
\tilde{\mathbb{E}}[\tilde{W}_1(t)|\mathcal{F}(s)] &= \frac{1}{Z(s)}\mathbb{E}[\tilde{W}_1(t)Z(t)|\mathcal{F}(s)] \\
&= \frac{1}{Z(s)}\tilde{W}_1(s)Z(s) = \tilde{W}_1(s).
\end{aligned}$$

Similarly for $\tilde{W}_2(t)$.

$\Rightarrow$ $\left\{\tilde{W}_i(t) : 0 \leq t \leq T\right\}$ is a martingale under $\tilde{\mathbb{P}}$. In other words, $\left\{\tilde{W}(t) = (\tilde{W}_1(t), \tilde{W}_2(t)) : 0 \leq t \leq T\right\}$ is a BM under $\tilde{\mathbb{P}}$.