Utrecht University
Mathematical Institute

Final Exam for Introduction to Financial Mathematics, WISB373

Wednesday June 29th 2022, 13:30-16:30 o'clock **(3 hours examination)**

1. Let $\{W(t) : t \geq 0\}$ be a Brownian motion, we define process $\{X(t) : t \geq 0\}$

$$X(t) = \frac{1}{\sqrt{3}} W(3t).$$

a. Prove that $\{X(t) : t \geq 0\}$ is a Brownian motion. (10 pts.)

b. Let $Y(t) = X^2(t) - 2\sqrt{c}t$ for some non-negative constant c and for all $t \geq 0$. For which value of c is the process $\{Y(t) : t \geq 0\}$ a martingale with respect to the filtration $\{\mathcal{F}(t) : t \geq 0\}$, with $\mathcal{F}(t) = \sigma(X(s) : s \leq t)$? (10 pts.)

2. Let $\{W(t) : 0 \leq t \leq T\}$ be a Brownian motion on probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\{\mathcal{F}(t) : 0 \leq t \leq T\}$ be its natural filtration, with $\mathcal{F} = \mathcal{F}(T)$. Consider a stock with price process $\{S(t) : 0 \leq t \leq T\}$, with

$$S(t) = S(0) \exp\left\{ \int_0^t \mathrm{e}^{-u} \mathrm{d}W(u) + \int_0^t (1 - \frac{1}{2}\mathrm{e}^{-2u}) \mathrm{d}u \right\}.$$

a. Let

$$X(t) = \int_0^t \mathrm{e}^{-u} \mathrm{d}W(u) + \int_0^t (1 - \frac{1}{2}\mathrm{e}^{-2u}) \mathrm{d}u$$

and determine the distribution of $X(t)$. (10 pts.)

b. Prove that $\{S(t) : t \geq 0\}$ is an Itô process. (10 pts.)

3. Let $\{W(t) : 0 \leq t \leq T\}$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\{\mathcal{F}(t) : 0 \leq t \leq T\}$ be its natural filtration, with $\mathcal{F} = \mathcal{F}(T)$. Consider a stock with price process $\{S(t) : 0 \leq t \leq T\}$ with $S(t) = t^4 + 4W(t)$.

a. Construct a measure $\tilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ (i.e., $\tilde{\mathbb{P}}(A) = 0$ if and only if $\mathbb{P}(A) = 0,\ A \in \mathcal{F}$), such that the price process $\{S(t) : 0 \leq t \leq T\}$ is a martingale under $\tilde{\mathbb{P}}$ and with respect to the filtration $\{\mathcal{F}(t) : 0 \leq t \leq T\}$. (10 pts.)

b. Consider a European call option on this stock with expiration date T and strike price K. Find an expression for

$$C(0) = \tilde{\mathbb{E}}[\mathrm{e}^{-rT}(S(T) - K)^+],$$

the price of this option at time 0, with constant interest rate r. (10 pts.)

Z.O.Z. Remaining questions on the other side.

4. Given a Radon-Nikodym derivative Z, and the associated Radon-Nikodym process $\{Z(t) : t \geq 0\}$, defined by $Z(t) = \mathbb{E}[Z|\mathcal{F}(t)]$, where $\{\mathcal{F}(t) : t \geq 0\}$ is a given filtration. We then have the change of probability measure, $\mathrm{d}\tilde{\mathbb{P}} = Z\mathrm{d}\mathbb{P}$, with the expectation under the $\tilde{\mathbb{P}}$-measure, i.e., $\tilde{\mathbb{E}}[Y] = \mathbb{E}[ZY]$

a. Let Y be a random variable which is $\mathcal{F}(t)$-measurable. Prove that

$$\mathbb{E}[YZ] = \tilde{\mathbb{E}}[Y] = \mathbb{E}[YZ(t)]. \quad \text{(10 pts.)}$$

b. Suppose Y is $\mathcal{F}(t)$-measurable, then prove (using partial averaging) that, for $s < t$

$$\tilde{\mathbb{E}}[Y|\mathcal{F}(s)] = \frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}(s)]. \quad \text{(10 pts.)}$$

5. Let $\{(W_1(t), W_2(t)) : t \geq 0\}$ be a 2-dimensional Brownian motion, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider two price processes $\{S_1(t) : t \geq 0\}$ and $\{S_2(t) : t \geq 0\}$ with corresponding SDE given by:

$$\begin{aligned} \mathrm{d}S_1(t) &= 2\alpha S_1(t)\mathrm{d}W_1(t) + 3\beta S_1(t)\mathrm{d}W_2(t) \\ \mathrm{d}S_2(t) &= \frac{\gamma}{2}S_2(t)\mathrm{d}t + \frac{\sigma}{4}S_2(t)\mathrm{d}W_1(t), \end{aligned}$$

where $\alpha, \beta, \gamma, \sigma$ are positive constants.

a. Show that $\{S_1(t)S_2(t) : t \geq 0\}$ is a 2-dimensional Itô-process. (10 pts.)

b. Consider a finite expiration date T, and suppose the interest rate is constant, $R(t) = r$ for all $t > 0$. Show that the market price equations have a unique solution, and determine the risk-neutral probability measure $\tilde{\mathbb{P}}$ for the process $\{(S_1(t), S_2(t) : 0 \leq t \leq T\}$. (10 pts.)

6. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\{W(t) = (W_1(t), W_2(t)) : t \geq 0\}$ a 2-dim. Brownian Motion with filtration $\{\mathcal{F}(t) : 0 \leq t \leq T\}$ (T is a fixed time). Let $\{\theta(t) = (\theta_1(t), \theta_2(t)) : 0 \leq t \leq T\}$ be an adapted process. Define,

$$Z(t) = \exp\left\{-\int_0^t \theta_1(u)\mathrm{d}W_1(u) - \int_0^t \theta_2(u)\mathrm{d}W_2(u) - \frac{1}{2}\int_0^t (\theta_1^2(u) + \theta_2^2(u))\mathrm{d}u\right\}$$

and $\tilde{W}_i(t) = W_i(t) + \int_0^t \theta_i(u)\mathrm{d}u), \quad i = 1, 2.$

Moreover, assume

$$\mathbb{E}\left[\int_0^t (\theta_1^2(u) + \theta_2^2(u))Z(u)\mathrm{d}u\right] < \infty,$$

set $Z = Z(T)$ and define $\tilde{\mathbb{P}}(A) = \int_A Z\mathrm{d}\mathbb{P}$.

Prove that $\{\tilde{W}(t) : 0 \leq t \leq T\}$ is a 2-dim. Brownian Motion under $\tilde{\mathbb{P}}$. (10 pts.)

Please, make sure that your name is written down on each of the submitted solutions.