1. Let \( \{W(t) : t \geq 0\} \) be a Brownian motion, we define process \( \{X(t) : t \geq 0\} \)

\[
X(t) = \frac{1}{\sqrt{3}} W(3t).
\]

a. Prove that \( \{X(t) : t \geq 0\} \) is a Brownian motion. (10 pts.)

b. Let \( Y(t) = X^2(t) - 2\sqrt{ct} \) for some non-negative constant \( c \) and for all \( t \geq 0 \). For which value of \( c \) is the process \( \{Y(t) : t \geq 0\} \) a martingale with respect to the filtration \( \{F(t) : t \geq 0\} \)? (10 pts.)

2. Let \( \{W(t) : 0 \leq t \leq T\} \) be a Brownian motion on probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \), and let \( \{F(t) : 0 \leq t \leq T\} \) be its natural filtration, with \( F = F(T) \). Consider a stock with price process \( \{S(t) : 0 \leq t \leq T\} \) with

\[
S(t) = S(0) \exp \left\{ \int_0^t e^{-u} dW(u) + \int_0^t (1 - \frac{1}{2}e^{-2u}) du \right\}.
\]

a. Let

\[
X(t) = \int_0^t e^{-u} dW(u) + \int_0^t (1 - \frac{1}{2}e^{-2u}) du
\]

and determine the distribution of \( X(t) \). (10 pts.)

b. Prove that \( \{S(t) : t \geq 0\} \) is an Itô process. (10 pts.)

3. Let \( \{W(t) : 0 \leq t \leq T\} \) be a Brownian motion on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \), and let \( \{F(t) : 0 \leq t \leq T\} \) be its natural filtration, with \( F = F(T) \). Consider a stock with price process \( \{S(t) : 0 \leq t \leq T\} \) with \( S(t) = t^4 + 4W(t) \).

a. Construct a measure \( \tilde{\mathbb{P}} \) equivalent to \( \mathbb{P} \) (i.e., \( \mathbb{P}(A) = 0 \) if and only if \( \tilde{\mathbb{P}}(A) = 0 \), \( A \in \mathcal{F} \) ), such that the price process \( \{S(t) : 0 \leq t \leq T\} \) is a martingale under \( \tilde{\mathbb{P}} \) and with respect to the filtration \( \{F(t) : 0 \leq t \leq T\} \). (10 pts.)

b. Consider a European call option on this stock with expiration date \( T \) and strike price \( K \). Find an expression for

\[
C(0) = \tilde{\mathbb{E}}[e^{-rT}(S(T) - K)^+],
\]

the price of this option at time 0, with constant interest rate \( r \). (10 pts.)

Z.O.Z. Remaining questions on the other side.
4. Given a Radon-Nikodym derivative $Z$, and the associated Radon-Nikodym process $\{Z(t) : t \geq 0\}$, defined by $Z(t) = \mathbb{E}[Z|\mathcal{F}(t)]$, where $\{\mathcal{F}(t) : t \geq 0\}$ is a given filtration. We then have the change of probability measure, $d\tilde{\mathbb{P}} = Zd\mathbb{P}$, with the expectation under the $\tilde{\mathbb{P}}$-measure, i.e., $\mathbb{E}[Y] = \mathbb{E}[ZY]
$

a. Let $Y$ be a random variable which is $\mathcal{F}(t)$-measurable. Prove that $\mathbb{E}[YZ] = \mathbb{E}[Y] = \mathbb{E}[YZ(t)]$. (10 pts.)

b. Suppose $Y$ is $\mathcal{F}(t)$-measurable, then prove (using partial averaging) that, for $s < t$
$$
\mathbb{E}[Y|\mathcal{F}(s)] = \frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}(s)].
$$
(10 pts.)

5. Let $\{(W_1(t), W_2(t)) : t \geq 0\}$ be a 2-dimensional Brownian motion, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider two price processes $\{S_1(t) : t \geq 0\}$ and $\{S_2(t) : t \geq 0\}$ with corresponding SDE given by:
$$
dS_1(t) = 2\alpha S_1(t)dW_1(t) + 3\beta S_1(t)dW_2(t)
$$
$$
dS_2(t) = \frac{\gamma}{2} S_2(t)dt + \sigma S_2(t)dW_1(t),
$$
where $\alpha, \beta, \gamma, \sigma$ are positive constants.

a. Show that $\{S_1(t)S_2(t) : t \geq 0\}$ is a 2-dimensional Itô-process. (10 pts.)

b. Consider a finite expiration date $T$, and suppose the interest rate is constant, $R(t) = r$ for all $t > 0$. Show that the market price equations have a unique solution, and determine the risk-neutral probability measure $\tilde{\mathbb{P}}$ for the process $\{(S_1(t), S_2(t)) : 0 \leq t \leq T\}$. (10 pts.)

6. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\{W(t) = (W_1(t), W_2(t)) : t \geq 0\}$ a 2-dim. Brownian Motion with filtration $\{\mathcal{F}(t) : 0 \leq t \leq T\}$ ($T$ is a fixed time). Let $\{\theta(t) = (\theta_1(t), \theta_2(t)) : 0 \leq t \leq T\}$ be an adapted process. Define,
$$
Z(t) = \exp \left\{ -\int_0^t \theta_1(u)dW_1(u) - \int_0^t \theta_2(u)dW_2(u) - \frac{1}{2} \int_0^t (\theta_1^2(u) + \theta_2^2(u))du \right\}
$$
and $\tilde{W}_i(t) = W_i(t) + \int_0^t \theta_i(u)du, \quad i = 1, 2.$
Moreover, assume
$$
\mathbb{E} \left[ \int_0^t (\theta_1^2(u) + \theta_2^2(u))Z(u)du \right] < \infty,
$$
set $Z = Z(T)$ and define $\tilde{\mathbb{P}}(A) = \int_A Zd\mathbb{P}$.
Prove that $\{\tilde{W}(t) : 0 \leq t \leq T\}$ is a 2-dim. Brownian Motion under $\tilde{\mathbb{P}}$. (10 pts.)

Please, make sure that your name is written down on each of the submitted solutions.