Utrecht University
Mathematical Institute

Solution for the Re-Examation for Introduction to Financial Mathematics, WISB373

Wednesday July 13th 2022 **(3 hours examination)**

1. Flip a biased coin three times with $\mathbb{P}(H) = \frac{1}{4}$ and $\mathbb{P}(T) = \frac{3}{4}$. So our probability space is $(\Omega, \mathcal{F}, \mathbb{P})$, with

$$\Omega = \{HHH; HHT, HTH, HTT, THH, THT, TTH, TTT\},$$

$\mathcal{F}$ is the power set of Ω, and

$$\begin{aligned} \mathbb{P}(HHH) &= \frac{1}{64}, \quad \mathbb{P}(TTT) = \frac{27}{64}. \\ \mathbb{P}(HHT) &= P(HTH) = P(THH) = \frac{3}{64}, \\ \mathbb{P}(HTT) &= P(THT) = P(TTH) = \frac{9}{64}. \end{aligned}$$

Let $\mathcal{F}_1$ be the σ-algebra containing the information on the first coin flip, i.e., $\mathcal{F}_1 = \sigma(\{A_H, A_T\})$, with $A_H = \{HHH, HHT, HTH, HTT\}$ and $A_T = \{THH, THT, TTH, TTT\}$. Define X on Ω by

$$X = 16 \cdot \mathbb{1}_{\{HHH;HHT\}} + 8 \cdot \mathbb{1}_{\{HTH,HTT,THH,THT\}} + 4 \cdot \mathbb{1}_{\{TTH,TTT\}}.$$

a. Find an explicit expression for $\mathbb{E}[X|\mathcal{F}_1]$. (10 pts.)

b. Define the price process S_0, S_1, S_2, S_3 on Ω by a tree, with $S_0 = 4$ and three coin tosses. Each time a head is tossed we have $S_i = 2S_{i-1}$, and each time a tail is obtained, we have $S_i = \frac{1}{2}S_{i-1}$.
Draw the corresponding tree, and show that $\sigma(S_2) \neq \mathcal{F}_2$. ($\mathcal{F}_2$ is the sigma algebra that contains the information about the first two coin flips.) (10 pts.)

Answers 1a. Since $\mathcal{F}_1$ is generated by the finite partition $\{A_H, A_T\}$, as we have seen in Homework 1,

$$\mathbb{E}[X|\mathcal{F}_1] = \frac{1}{\mathbb{P}(A_H)} \mathbb{1}_{A_H} \mathbb{E}[\mathbb{1}_{A_H} X] + \frac{1}{\mathbb{P}(A_T)} \mathbb{1}_{A_T} \mathbb{E}[\mathbb{1}_{A_T} X].$$

Now,

$$\mathbb{1}_{A_H} X = 16 \cdot \mathbb{1}_{\{HHH,HHT\}} + 8 \cdot \mathbb{1}_{\{HTH,HTT\}},$$

and

$$\mathbb{1}_{A_T} X = 8 \cdot \mathbb{1}_{\{THH,THT\}} + 4 \cdot \mathbb{1}_{\{TTH,TTT\}}.$$

Thus,

$$\frac{1}{\mathbb{P}(A_H)} \mathbb{1}_{A_H} \mathbb{E}[\mathbb{1}_{A_H} X] = 4(16 \cdot \mathbb{P}\{HHH; HHT\} + 8 \cdot \mathbb{P}\{HTH, HTT\})$$

and

$$\frac{1}{\mathbb{P}(A_T)} \mathbb{1}_{A_T} \mathbb{E}[\mathbb{1}_{A_T} X] = \frac{4}{3}(8 \cdot \mathbb{P}\{THH, THT\} + 4 \cdot \mathbb{P}\{TTH, TTT\}).$$

Therefore,

$$\mathbb{E}[X|\mathcal{F}_1] = 10\mathbb{1}_{A_H} + 5\mathbb{1}_{A_T}.$$

Answer 1b. Flip a coin 3 times. Associated with this, we have the filtration $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{F}_3$, that we have seen before:

$$\begin{aligned}\mathcal{F}_1 &= \{\emptyset, \Omega, A_H, A_T\}\\ \mathcal{F}_2 &= \{\emptyset, \Omega, A_H, A_T, A_{HH}, A_{TT}, \ldots\}\\ \mathcal{F}_3 &= \mathcal{P}(\Omega)\end{aligned}$$

Define S_0, S_1, S_2, S_3 on Ω by a tree:

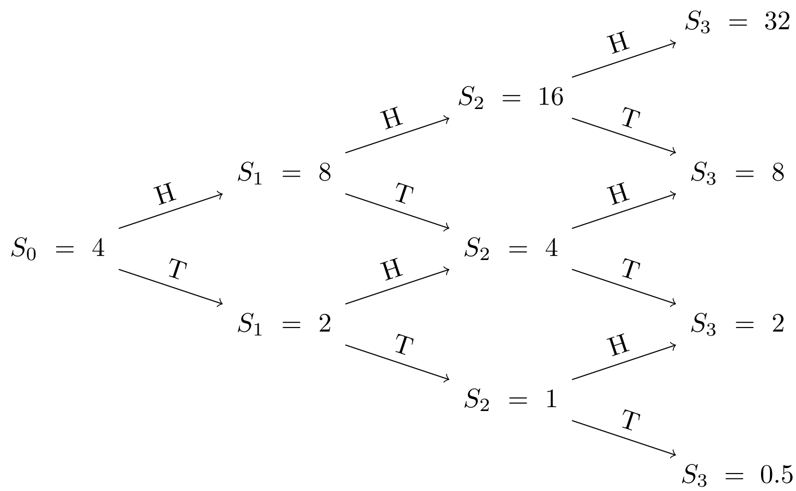

Figure 1: Tree for S_0, S_1, S_2, S_3 on Ω.

Consider S_2: To find $\sigma(S_2)$, it is enough to start with the sets $\{S_2 = 16\}, \{S_2 = 4\}, \{S_2 = 1\}$.
$\{S_2 = 16\} = A_{HH}, \{S_2 = 4\} = A_{HT} \bigcup A_{TH}, \{S_2 = 1\} = A_{TT}$.

So,

$$\begin{aligned}\sigma(S_2) &= \{\emptyset, \Omega, A_{HH}, A_{HT} \bigcup A_{TH}, A_{TT},\\ &\quad A_{HH}^c = A_{HT} \bigcup A_{TH} \bigcup A_{TT},\\ &\quad A_{HH} \bigcup A_{TT} = (A_{HT} \bigcup A_{TH})^c,\\ &\quad A_{TT}^c = A_{HH} \bigcup A_{HT} \bigcup A_{TH}\} \subset \mathcal{F}_2.\end{aligned}$$

$A_{HT} \in \mathcal{F}_2$, however, $A_{HT} \notin \sigma(S_2)$.
Since $\mathcal{F}_2$ is finer, if we know in which element of $\mathcal{F}_2$ our outcome lies, then we know the value of S_2. The same, of course, is true for $\sigma(S_2)$.

2. Let $\{M(t) : t \geq 0\}$ be a martingale with respect to the filtration $\{\mathcal{F}(t) : t \geq 0\}$. Assume $\{M(t) : t \geq 0\}$ has continuous paths, $M(0) = 0$, and the Quadratic Variation $[M, M](t) = t, t \geq 0$. Show that $M(t) \sim N(0, t)$ by showing, for any t, $\mathbb{E}[\mathrm{e}^{uM(t)}] = \mathrm{e}^{\frac{1}{2}u^2 t}$. (10 pts.)

Answer 2. The book proves that $M(t) \sim N(0, t)$ by showing, for any t, $\mathbb{E}[\mathrm{e}^{uM(t)}] = \mathrm{e}^{\frac{1}{2}u^2 t}$, equivalently

$$\mathbb{E}[\mathrm{e}^{uM(t) - \frac{1}{2}u^2 t}] = 1.$$

We have seen this early in the book when $M(t)$ was $W(t)$.

$\Rightarrow$ We need to show that $e^{uM(t)-\frac{1}{2}u^2t}$ is a martingale.

Note that the expression $e^{uM(t)-\frac{1}{2}u^2t}$, resembles a Geometric Brownian Motion, where the BM $W(t)$ is replaced by the martingale $M(t)$, so we can use Itô-Doeblin with $f(t,x) = e^{ux-\frac{1}{2}u^2t}$.

We can do because basically the only properties used to define the Itô-integral w.r.t. BM $\{W(t) : t \geq 0\}$, were

(1) Continuous paths
(2) Quadratic variation $[W,W](t) = t$
(3) Martingale property $\to$ leads to $I(t)$ being a martingale

So, one can repeat the construction of the Itô-integral, but now w.r.t. a martingale $\{M(t) : t \geq 0\}$, satisfying properties (1),(2) and (3). Which allows us in a similar way, the notion of an Itô-integral $\int_0^t \Delta(t)(t)$ w.r.t. a martingale.

$\Rightarrow$ The resulting Itô-integral $\int_0^t \Delta(u)(u)$ is a martingale and $\mathbb{E}[\int_0^t \Delta(u)(u)] = 0$!

With the aim to show

$$\mathbb{E}[e^{uM(t)-\frac{1}{2}u^2t}] = 1,$$

we apply the Itô-Doeblin formula, with

$$f(t,x) = e^{ux-\frac{1}{2}u^2t}.$$

We then have

$$\begin{aligned}
f_t(t,x) &= -\frac{1}{2}u^2 e^{ux-\frac{1}{2}u^2t} = -\frac{1}{2}u^2 f(t,x) \\
f_x(t,x) &= u e^{ux-\frac{1}{2}u^2t} = uf(t,x) \\
f_{xx}(t,x) &= u^2 e^{ux-\frac{1}{2}u^2t} = u^2 f(t,x). \text{ Then:} \\
e^{uM(t)-\frac{1}{2}u^2t} &= f(t,M(t)) = f(0,M(0) + \int_0^t -\frac{1}{2}u^2 f(s,M(s)) \\
&+ \int_0^t uf(s,M(s))(s) + \frac{1}{2}\int_0^t u^2 f(s,M(s))(s)(s)
\end{aligned}$$

This leads to:

$$e^{uM(t)-\frac{1}{2}u^2t} = f(t,M(t)) = 1 + \int_0^t uf(s,M(s))(s)$$

where the RHS contains an Itô-integral and is hence a martingale. So,

$$\mathbb{E}[e^{uM(t)-\frac{1}{2}u^2t}] = 1 + \mathbb{E}\left[\int_0^t uf(s,M(s))(s)\right] = 1,$$

as required (integral equals 0).

In fact, we have shown that $e^{uM(t)-\frac{1}{2}u^2t}$ is a martingale.

So, $M(t) \sim N(0,t)$. Now, let $s < t$, from the above we have

$$\begin{aligned}
\mathbb{E}[e^{uM(t)-\frac{1}{2}u^2t}|\mathcal{F}(s)] &= e^{uM(s)-\frac{1}{2}u^2s} \\
\Rightarrow \mathbb{E}[e^{u(M(t)-M(s))}|\mathcal{F}(s)] &= e^{\frac{1}{2}u^2(t-s)} \qquad (1) \\
\Rightarrow \mathbb{E}[e^{u(M(t)-M(s))}] &= \mathbb{E}[\mathbb{E}[e^{u(M(t)-M(s))}|\mathcal{F}(s)]] = e^{\frac{1}{2}u^2(t-s)} \\
\Rightarrow M(t)-M(s) &\sim N(0,t-s) \text{ and Eq.(1):}
\end{aligned}$$

$\Rightarrow \mathbb{E}[e^{uM(t)}|\mathcal{F}(s)] = e^{uM(s)}e^{\frac{1}{2}u^2(t-s)}(2)$

3. Show that, for a continuously, differentiable function $g(t)$, the process

$$X(t) = g(t)W(t) - \int_0^t g'(z)W(z)\mathrm{d}z,$$

is a martingale w.r.t. the natural filtration generated by the Brownian motion $W(t)$, where $g'(t)$ is the first derivative of $g(t)$, and subsequently show that

$$\mathbb{E}[\mathrm{e}^{2t}W(t)] = \mathbb{E}\left[\int_0^t 2\mathrm{e}^{2z}W(z)\mathrm{d}z\right].$$

(10 pts.)

Answer 3. The corresponding SDE must not contain a drift term, if it represents a martingale process. The differential process of the integral equation is found to be,

$$\begin{aligned}\mathrm{d}X(t) &= d(g(t)W(t)) - g'(t)W(t)\mathrm{d}t\\ &= g'(t)W(t)\mathrm{d}t - g(t)\mathrm{d}W(t) - g'(t)W(t)\mathrm{d}t\\ &= -g(t)\mathrm{d}W(t),\end{aligned}$$

where the differentiation, denoted by d, is in the Itô sense. Hence, the process is a martingale, as we don't encounter any dt-term.

Express the term $e^{2t}W(t)$ in its integral form. as follows,

$$\begin{aligned}d(e^{2t}W(t)) &= 2e^{2t}W(t)\mathrm{d}t + e^{2t}\mathrm{d}W(t)\\ e^{2t}W(t) &= \int_t^0 (2e^{2u}W(u) + e^{2u}\mathrm{d}W(u)),\end{aligned}$$

where $W(0) = 0$ is used in the second equation. Taking the expectation at both sides and using $\mathbb{E}[\int_0^t e^{2u}\mathrm{d}W(u)] = 0$, because infinitesimal increments of Brownian motion are governed by a normal distribution with zero mean. Therefore,

$$\mathbb{E}[e^{2t}W(t)] = \mathbb{E}\big[\int_0^t 2e^{2u}W(u)\mathrm{d}t\big].$$

4. In the Vasicek interest rate model, the interest rate process $\{R(t) : t \geq 0\}$ satisfies the SDE, given by

$$R(t) = R(0) + \int_0^t (\alpha - \beta R(u)) + \int_0^t \sigma\mathrm{d}W(u), \tag{3}$$

where $\alpha, \beta, \sigma > 0$ are constants.

We have the following closed-form solution for $R(t)$.

$$R(t) = \mathrm{e}^{-\beta t}R(0) + \frac{\alpha}{\beta}(1 - \mathrm{e}^{-\beta t}) + \sigma\mathrm{e}^{-\beta t}\int_0^t \mathrm{e}^{\beta s}\mathrm{d}W(s). \tag{4}$$

a. Determine the distribution of

$$X(t) = \int_0^t \mathrm{e}^{\beta s}\mathrm{d}W(s),$$

and subsequently determine the distribution of $R(t)$. (10 pts.)

b. Show that Equation (5) satisfies Equation (3). (10 pts.)

c. Explain why $R(t)$ is called a mean reverting process. (10 pts.)

Answer 4a. In this model,

$$\mathrm{d}R(t) = (\alpha - \beta R(t))\mathrm{d}t + \sigma \mathrm{d}W(t),$$

and our aim is to find a *closed form solution* for $R(t)$.
With,

$$R(t) = \mathrm{e}^{-\beta t}R(0) + \frac{\alpha}{\beta}(1-\mathrm{e}^{-\beta t}) + \sigma \mathrm{e}^{-\beta t}\int_0^t \mathrm{e}^{\beta s}\mathrm{d}W(s). \qquad (5)$$

the RHS does not contain $R(t)$.
The claim is proved by showing that (5) satisfies (3) by using the Itô-Doeblin formula.
$X(t)$ is an Itô-integral of the deterministic process $\Delta(t) = \mathrm{e}^{\beta t}$)
So, we have a Theorem which states:

$$X(t) \sim N(0, \int_0^t \mathrm{e}^{2\beta s}) = N(0, \frac{1}{2\beta}(\mathrm{e}^{2\beta t} - 1)).$$

Answer 4b. Note that the answer that we are seeking has the form $R(t) = f(t, X(t))$, so we apply Itô-Doeblin for Itô processes, but first we calculate the necessary ingredients

$$\begin{aligned}
f_t(t,x) &= -\beta \mathrm{e}^{-\beta t}R(0) + \alpha \mathrm{e}^{-\beta t} - \sigma\beta \mathrm{e}^{-\beta t}x \pm \frac{\alpha\beta}{\beta} \\
&= -\beta\left[\mathrm{e}^{-\beta t}R(0) + \frac{\alpha}{\beta}(1-\mathrm{e}^{-\beta t}) + \sigma \mathrm{e}^{-\beta t}x\right] + \alpha \\
&= \alpha - \beta f(t,x) \\
f_x(t,x) &= \sigma \mathrm{e}^{-\beta t} \text{ and } f_{xx}(t,x) = 0
\end{aligned}$$

So, using the Itô-Doeblin formula $R(t) =$

$$\begin{aligned}
f(t,X(t)) &= f(0,X(0)) + \int_0^t (\alpha - \beta f(u,X(u))) + \int_0^t \sigma \mathrm{e}^{-\beta u}\mathrm{d}X(u) \\
&= R(0) + \int_0^t (\alpha - \beta f(u,X(u))) + \int_0^t \sigma \mathrm{e}^{-\beta u}\mathrm{e}^{\beta u}\mathrm{d}W(u)
\end{aligned}$$

Thus,

$$R(t) = \mathrm{e}^{-\beta t}R(0) + \frac{\alpha}{\beta}(1-\mathrm{e}^{-\beta t}) + \sigma \mathrm{e}^{-\beta t}\int_0^t \mathrm{e}^{\beta s}\mathrm{d}W(s)$$

is a solution to (3).

We can in fact determine the distribution of $R(t)$: Note that

$$R(t) = a(t) + b(t)X(t),$$

with $a(t), b(t)$ non-random and

$$X(t) \sim N\left(0, \int_0^t \mathrm{e}^{2\beta s}\right).$$

In fact, $a(t) = \mathrm{e}^{-\beta t}R(0) + \frac{\alpha}{\beta}(1 - \mathrm{e}^{-\beta t})$ and $b(t) = \sigma \mathrm{e}^{-\beta t}$. Hence

$$R(t) \sim N\left(\mathrm{e}^{-\beta t}R(0) + \frac{\alpha}{\beta}(1 - \mathrm{e}^{-\beta t}), \sigma^2 \mathrm{e}^{-2\beta t} \cdot \frac{1}{2\beta}(\mathrm{e}^{2\beta t} - 1)\right)$$

$$R(t) = N(a(t), b^2(t)\mathbb{V}\mathrm{ar}(X(t)))$$

Answer 4c. $R(t)$ is *mean reverting*, which implies the following:

$$(t) = (\alpha - \beta R(t))\mathrm{d}t + \sigma \mathrm{d}W(t)$$

* If $R(t) = \frac{\alpha}{\beta}$, then $\mathrm{d}t$-term is 0
* If $R(t) > \frac{\alpha}{\beta}$, then $\mathrm{d}t$-term is < 0
* If $R(t) < \frac{\alpha}{\beta}$, then $\mathrm{d}t$-term is > 0.

5. Let $W(t) = (W_1(t), W_2(t))$ be a 2-dim. BM (so, by definition, $W_1(t)$ and $W_2(t)$ are independent), defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

a. Let $W_3(t) = \rho \mathrm{d}W_1(t) + \sqrt{1-\rho^2}\mathrm{d}W_2(t)$. Show that $W_3(t)$ is a Brownian Motion by Lévy's characterization. (10 pts.)

b. Consider the (price) processes:

$$\begin{aligned} \mathrm{d}S_1(t) &= \alpha_1 S_1(t)\mathrm{d}t + \sigma S_1(t)\mathrm{d}W_1(t) \\ \mathrm{d}S_2(t) &= rS_2(t)\mathrm{d}t + 0.1S_2(t)\mathrm{d}W_3(t), \end{aligned}$$

where $r, \alpha_1, \sigma_1, \sigma_2 > 0$ and $-1 \leq \rho \leq 1$ are constants. Determine the correlation between $S_1(t)$ and $S_2(t)$. (10 pts.)

c. Show that $\{S_1(t)S_2(t) : t \geq 0\}$ is a 2-dimensional Itô-process. (10 pts.)

d. Consider a finite expiration date T, and a constant interest rate, $R(t) = r$, for all $t > 0$. Show that the market price equations have a unique solution, and determine the risk-neutral probability measure $\tilde{\mathbb{P}}$ for the process $\{(S_1(t), S_2(t) : 0 \leq t \leq T\}$. (10 pts.)

Answer to 5a. Let $W(t) = (W_1(t), W_2(t))$ be a 2-dim. BM (so, by definition, $W_1(t)$ and $W_2(t)$ are independent). Let

$$W_3(t) = \rho \mathrm{d}W_1(t) + \sqrt{1-\rho^2}\mathrm{d}W_2(t).$$

Then $\{W_3(t) : t \geq 0\}$ is a martingale, with $W_3(0) = 0$, and

$$\mathrm{d}W_3(t)\mathrm{d}W_3(t) = \rho^2 \mathrm{d}t + (1-\rho^2)\mathrm{d}t = \mathrm{d}t.$$

Furthermore, it has continuous paths.

So, by Lévy's characterization in one-dimension, we see that $\{W_3(t) : t \geq 0\}$ is a BM, and we can write $\{S_2(t) : t \geq 0\}$ as

$$\mathrm{d}S_2(t) = \alpha_2 S_2(t)\mathrm{d}t + \sigma_2 S_2(t)\mathrm{d}W_3(t)$$

so that $(S_2(t) : t \geq 0\}$ is also a GBM of the form:

$$S_2(t) = S_2(0)\exp\{\sigma_2 W_3(t) + (\alpha_2 - \frac{1}{2}\sigma_2^2)t\}.$$

Answer 5b. We first use the Itô-product rule, to write

$$\begin{aligned} \mathrm{d}(W_1(t)W_3(t)) &= W_1(t)\mathrm{d}W_3(t) + W_3(t)\mathrm{d}W_1(t) + \mathrm{d}W_1(t)\mathrm{d}W_3(t) \\ & \quad \text{write out} \quad W_3(t)) \\ &= W_1(t)\mathrm{d}W_3(t) + W_3(t)\mathrm{d}W_1(t) + \rho\mathrm{d}t \\ & \quad \text{and integral form} \\ W_1(t)W_3(t) &= W_1(0)W_3(0) + \int_0^t W_1(t)\mathrm{d}W_3(t) \\ &+ \int_0^t W_3(t)\mathrm{d}W_1(t) + \rho t \\ \Rightarrow & \quad \mathbb{E}[W_1(t)W_3(t)] = \rho t = (W_1(t), W_3(t)) \end{aligned}$$

Answer to 5c. We write the processes as

$$\begin{aligned} \mathrm{d}S_1(t) &= \alpha_1 S_1(t)\mathrm{d}t + \sigma_1 S_1(t)\mathrm{d}W_1(t) \\ \mathrm{d}S_2(t) &= \alpha_2 S_2(t)\mathrm{d}t + \sigma_2 S_2(t)[\rho\mathrm{d}W_1(t) + \sqrt{1-\rho^2}\mathrm{d}W_2(t)] \end{aligned}$$

Please, make sure that your name is written down on each of the submitted solutions.