Utrecht University
Mathematical Institute

Re-Examation for Introduction to Financial Mathematics, WISB373

Wednesday July 13th 2022, 13:30-16:30 o'clock **(3 hours examination)**

1. Flip a biased coin three times with $\mathbb{P}(H) = \frac{1}{4}$ and $\mathbb{P}(T) = \frac{3}{4}$. So our probability space is $(\Omega, \mathcal{F}, \mathbb{P})$, with

$$\Omega = \{HHH; HHT, HTH, HTT, THH, THT, TTH, TTT\},$$

$\mathcal{F}$ is the power set of Ω, and

$$\begin{aligned} \mathbb{P}(HHH) &= \frac{1}{64}, \quad \mathbb{P}(TTT) = \frac{27}{64}. \\ \mathbb{P}(HHT) &= P(HTH) = P(THH) = \frac{3}{64}, \\ \mathbb{P}(HTT) &= P(THT) = P(TTH) = \frac{9}{64}. \end{aligned}$$

Let $\mathcal{F}_1$ be the σ-algebra containing the information on the first coin flip, i.e., $\mathcal{F}_1 = \sigma(\{A_H, A_T\})$, with $A_H = \{HHH, HHT, HTH, HTT\}$ and $A_T = \{THH, THT, TTH, TTT\}$. Define X on Ω by

$$X = 16 \cdot \mathbb{1}_{\{HHH;HHT\}} + 8 \cdot \mathbb{1}_{\{HTH,HTT,THH,THT\}} + 4 \cdot \mathbb{1}_{\{TTH,TTT\}}.$$

- **a.** Find an explicit expression for $\mathbb{E}[X|\mathcal{F}_1]$. (10 pts.)
- **b.** Define the price process S_0, S_1, S_2, S_3 on Ω by a tree, with $S_0 = 4$ and three coin tosses. Each time a head is tossed we have $S_i = 2S_{i-1}$, and each time a tail is obtained, we have $S_i = \frac{1}{2}S_{i-1}$.
  Draw the corresponding tree, and show that $\sigma(S_2) \neq \mathcal{F}_2$. ($\mathcal{F}_2$ is the sigma algebra that contains the information about the first two coin flips.) (10 pts.)

2. Let $\{M(t) : t \geq 0\}$ be a martingale with respect to the filtration $\{\mathcal{F}(t) : t \geq 0\}$. Assume $\{M(t) : t \geq 0\}$ has continuous paths, $M(0) = 0$, and the Quadratic Variation $[M, M](t) = t, t \geq 0$. Show that $M(t) \sim N(0, t)$ by showing, for any t, $\mathbb{E}[\mathrm{e}^{uM(t)}] = \mathrm{e}^{\frac{1}{2}u^2 t}$. (10 pts.)

3. Show that, for a continuously, differentiable function $g(t)$, the process

$$X(t) = g(t)W(t) - \int_0^t g'(z)W(z)\mathrm{d}z,$$

is a martingale w.r.t. the natural filtration generated by the Brownian motion $W(t)$, where $g'(t)$ is the first derivative of $g(t)$, and subsequently show that

$$\mathbb{E}[\mathrm{e}^{2t}W(t)] = \mathbb{E}\left[\int_0^t 2\mathrm{e}^{2z}W(z)\mathrm{d}z\right].$$

(10 pts.)

Z.O.Z. Remaining questions on the other side.

4. In the Vasicek interest rate model, the interest rate process $\{R(t) : t \geq 0\}$ satisfies the SDE, given by

$$R(t) = R(0) + \int_0^t (\alpha - \beta R(u)) + \int_0^t \sigma \mathrm{d}W(u), \tag{1}$$

where $\alpha, \beta, \sigma > 0$ are constants.

We have the following closed-form solution for $R(t)$.

$$R(t) = \mathrm{e}^{-\beta t} R(0) + \frac{\alpha}{\beta}(1 - \mathrm{e}^{-\beta t}) + \sigma \mathrm{e}^{-\beta t} \int_0^t \mathrm{e}^{\beta s} \mathrm{d}W(s). \tag{2}$$

a. Determine the distribution of

$$X(t) = \int_0^t \mathrm{e}^{\beta s} \mathrm{d}W(s),$$

and subsequently determine the distribution of $R(t)$. (10 pts.)

b. Show that Equation (2) satisfies Equation (1). (10 pts.)

c. Explain why $R(t)$ is called a mean reverting process. (10 pts.)

5. Let $W(t) = (W_1(t), W_2(t))$ be a 2-dim. BM (so, by definition, $W_1(t)$ and $W_2(t)$ are independent), defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

a. Let $W_3(t) = \rho \mathrm{d}W_1(t) + \sqrt{1 - \rho^2} \mathrm{d}W_2(t)$. Show that $W_3(t)$ is a Brownian Motion by Lévy's characterization. (10 pts.)

b. Consider the (price) processes:

$$\begin{aligned} \mathrm{d}S_1(t) &= \alpha_1 S_1(t) \mathrm{d}t + \sigma S_1(t) \mathrm{d}W_1(t) \\ \mathrm{d}S_2(t) &= r S_2(t) \mathrm{d}t + 0.1 S_2(t) \mathrm{d}W_3(t), \end{aligned}$$

where $r, \alpha_1, \sigma_1, \sigma_2 > 0$ and $-1 \leq \rho \leq 1$ are constants. Determine the correlation between $S_1(t)$ and $S_2(t)$. (10 pts.)

c. Show that $\{S_1(t) S_2(t) : t \geq 0\}$ is a 2-dimensional Itô-process. (10 pts.)

d. Consider a finite expiration date T, and a constant interest rate, $R(t) = r$, for all $t > 0$. Show that the market price equations have a unique solution, and determine the risk-neutral probability measure $\tilde{\mathbb{P}}$ for the process $\{(S_1(t), S_2(t) : 0 \leq t \leq T\}$. (10 pts.)

Please, make sure that your name is written down on each of the submitted solutions.