1. Flip a biased coin three times with $P(H) = \frac{1}{4}$ and $P(T) = \frac{3}{4}$. So our probability space is $(\Omega, \mathcal{F}, \mathbb{P})$, with

$$\Omega = \{HHH; HHT, HTH, HTT, THH, THT, TTH, TTT\},$$

$\mathcal{F}$ is the power set of $\Omega$, and

$$\mathbb{P}(HHH) = \frac{1}{64}, \quad \mathbb{P}(TTT) = \frac{27}{64},$$

$$\mathbb{P}(HHT) = \mathbb{P}(HTH) = \mathbb{P}(THH) = \frac{3}{64},$$

$$\mathbb{P}(HTT) = \mathbb{P}(THT) = \mathbb{P}(TTH) = \frac{9}{64}.$$

Let $\mathcal{F}_1$ be the $\sigma$-algebra containing the information on the first coin flip, i.e., $\mathcal{F}_1 = \sigma(\{A_H, A_T\})$, with $A_H = \{HHH, HHT, HTH, HTT\}$ and $A_T = \{THH, THT, TTH, TTT\}$. Define $X$ on $\Omega$ by

$$X = 16 \cdot 1_{\{HHH,HHT\}} + 8 \cdot 1_{\{HTH,HTT,THH,THT\}} + 4 \cdot 1_{\{TTH,TTT\}}.$$

a. Find an explicit expression for $E[X|\mathcal{F}_1]$. (10 pts.)

b. Define the price process $S_0, S_1, S_2, S_3$ on $\Omega$ by a tree, with $S_0 = 4$ and three coin tosses. Each time a head is tossed we have $S_i = 2S_{i-1}$, and each time a tail is obtained, we have $S_i = \frac{1}{2}S_{i-1}$.

Draw the corresponding tree, and show that $\sigma(S_2) \neq \mathcal{F}_2$. (10 pts.)

2. Let $\{M(t) : t \geq 0\}$ be a martingale with respect to the filtration $\{\mathcal{F}(t) : t \geq 0\}$. Assume $\{M(t) : t \geq 0\}$ has continuous paths, $M(0) = 0$, and the Quadratic Variation $[M, M](t) = t, t \geq 0$. Show that $M(t) \sim N(0, t)$ by showing, for any $t$, $E[e^{uM(t)}] = e^{\frac{1}{2}u^2t}$. (10 pts.)

3. Show that, for a continuously, differentiable function $g(t)$, the process

$$X(t) = g(t) W(t) - \int_0^t g'(z) W(z) dz,$$

is a martingale w.r.t. the natural filtration generated by the Brownian motion $W(t)$, where $g'(t)$ is the first derivative of $g(t)$, and subsequently show that

$$E[e^{2t} W(t)] = E\left[\int_0^t 2e^{2z} W(z) dz\right].$$

(10 pts.)

Z.O.Z. Remaining questions on the other side.
4. In the Vasicek interest rate model, the interest rate process \( \{ R(t) : t \geq 0 \} \) satisfies the SDE, given by

\[
R(t) = R(0) + \int_0^t (\alpha - \beta R(u)) + \int_0^t \sigma dW(u),
\]

where \( \alpha, \beta, \sigma > 0 \) are constants.

We have the following closed-form solution for \( R(t) \).

\[
R(t) = e^{-\beta t} R(0) + \frac{\alpha}{\beta} (1 - e^{-\beta t}) + \sigma e^{-\beta t} \int_0^t e^{\beta s} dW(s).
\]

a. Determine the distribution of \( X(t) = \int_0^t e^{\beta s} dW(s) \), and subsequently determine the distribution of \( R(t) \). (10 pts.)

b. Show that Equation (2) satisfies Equation (1). (10 pts.)

c. Explain why \( R(t) \) is called a mean reverting process. (10 pts.)

5. Let \( W(t) = (W_1(t), W_2(t)) \) be a 2-dim. BM (so, by definition, \( W_1(t) \) and \( W_2(t) \) are independent), defined on a probability space \( (\Omega, \mathcal{F}, P) \).

a. Let \( W_3(t) = \rho dW_1(t) + \sqrt{1 - \rho^2} dW_2(t) \). Show that \( W_3(t) \) is a Brownian Motion by Lévy’s characterization. (10 pts.)

b. Consider the (price) processes:

\[
\begin{align*}
\text{d} S_1(t) &= \alpha_1 S_1(t) \text{d} t + \sigma S_1(t) \text{d} W_1(t) \\
\text{d} S_2(t) &= r S_2(t) \text{d} t + 0.1 S_2(t) \text{d} W_3(t),
\end{align*}
\]

where \( r, \alpha_1, \sigma_1, \sigma_2 > 0 \) and \(-1 \leq \rho \leq 1\) are constants. Determine the correlation between \( S_1(t) \) and \( S_2(t) \). (10 pts.)

c. Show that \( \{ S_1(t), S_2(t) : t \geq 0 \} \) is a 2-dimensional Itô-process. (10 pts.)

d. Consider a finite expiration date \( T \), and a constant interest rate, \( R(t) = r \), for all \( t > 0 \). Show that the market price equations have a unique solution, and determine the risk-neutral probability measure \( \tilde{P} \) for the process \( \{(S_1(t), S_2(t) : 0 \leq t \leq T) \}. \) (10 pts.)

Please, make sure that your name is written down on each of the submitted solutions.