Utrecht University
Mathematical Institute

Solutions to Mid-Term Exam for Introduction to Financial Mathematics, WISB373

Friday May 20th 2022, 9:00 - 11:00 **(2 hours examination)**

For each of the eight exercises 10 points can be obtained.

1. For any integrable random variable X and any event $B \in \mathcal{F}$ such that $\mathbb{P}(B) \neq 0$, the conditional expectation of X given B is defined by

$$E[X|B] = \frac{1}{\mathbb{P}(B)} \int_B X \mathrm{d}\mathbb{P}.$$

a. Show that if

$$X(\omega) = \mathbb{1}_A(\omega) = \begin{cases} 1 & \omega \in A, \\ 0 & \omega \notin A, \end{cases}$$

then $\mathbb{E}[\mathbb{1}_A|B] = \mathbb{P}(A|B)$, where

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

is the conditional probability of A given B.
Furthermore, show that $\mathbb{E}[X|\Omega] = \mathbb{E}[X]$.

b. If X and Y are random variables and $\mathbb{E}[Y|X] = c$, then show that X and Y are uncorrelated. (Hint: It's sufficient to show that $\mathrm{Cov(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 0}$.)

c. Three coins, 10 cents, 20 cents and 50 cents are tossed. The values of those coins that land with heads up are added to give us the total amount X. What is the expected total amount X given that two coins have landed with heads up?

Answers:

a. By definition, $\mathbb{E}[\mathbb{1}_A|B] = \frac{1}{\mathbb{P}(B)} \int_{A \cap B} \mathrm{d}\mathbb{P} = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} := \mathbb{P}(A|B)$.
We also show that $\mathbb{E}[X|\Omega] = \mathbb{E}[X]$. Since $\mathbb{P}(\Omega) = 1$ and $\int_\Omega X \mathrm{d}\mathbb{P} = \mathbb{E}[X]$, it follows that

$$\mathbb{E}[X|\Omega] = \frac{1}{\mathbb{P}(\Omega)} \int_\Omega X \mathrm{d}\mathbb{P} = \mathbb{E}[X].$$

b. We will show that $\mathrm{Cov(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 0}$..
We have: $\mathbb{E}[XY] = \mathbb{E}[\mathbb{E}[XY|X]] = \mathbb{E}[X\mathbb{E}[Y|X]] = c\mathbb{E}[X]$. Moreover, $\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|X]] = c$ and $\mathrm{Cov[X, Y] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = c\mathbb{E}[X] - c\mathbb{E}[X]] = 0}$.

c. Let B denote the event that two coins have landed heads up. We want to find $\mathbb{E}[Z|B]$. B consists of three elements, with $B = \{HHT, HTH, THH\}$. each having probability 1/8. The corresponding values of Z are:

$$\begin{aligned} Z(HHT) &= 10 + 20 = 30, \\ Z(HTH) &= 10 + 50 = 60, \\ Z(THH) &= 20 + 50 = 70. \end{aligned}$$

Therefore,

$$\mathbb{E}[Z|B] = \frac{1}{\mathbb{P}(B)} \int_B Z \mathrm{d}\mathbb{P} = \frac{1}{3/8} \left(\frac{30}{8} + \frac{60}{8} + \frac{70}{8} \right) = 53\frac{1}{3}.$$

2. Determine whether the following random variables are a martingale with respect to filtration $\mathcal{F}_t$ and give arguments for your statement. *Answers:* The processes are adapted to the filtration, because $W(t)$ is.

a. The requested form is not a martingale, however, $\mathrm{e}^{W(t)}\mathrm{e}^{-\frac{(t)}{2}}$ is a martingale with respect to the filtration $\mathcal{F}(t)$. We will show that this is a martingale (at the same time thus showing that the requested form is not): First of all, $\mathrm{e}^{W(t)}\mathrm{e}^{-\frac{(t)}{2}}$ is measurable. Secondly: Let $0 < s < t$, because $W(t) - W(s)$ is independent of $\mathcal{F}(s)$, and $W(s)$ is $\mathcal{F}(s)$-measurable, we find

$$\begin{aligned} \mathbb{E}[\mathrm{e}^{W(t)}|\mathcal{F}(s)] &= \mathbb{E}[\mathrm{e}^{W(t)-W(s)}\mathrm{e}^{W(s)}|\mathcal{F}(s)] \\ &= \mathrm{e}^{W(s)}\mathbb{E}[\mathrm{e}^{W(t)-W(s)}|\mathcal{F}(s)] = \mathrm{e}^{W(s)}\mathbb{E}[\mathrm{e}^{W(t)-W(s)}]. \end{aligned}$$

The increment $W(t) - W(s)$ has the normal distribution with mean 0 and variance $t - s$. And the expectation of $\mathrm{e}^{W(t)-W(s)}$, think of the moment generating function, equals $\mathrm{e}^{\frac{1}{2}(t-s)}$. So, the requested stochastic variable, which is different from the one we just considered, is **not** a martingale.

b. Again, the requested expression is not a martingale. However, this form, $Z(t) = |W(t)|^2 - t$, is a martingale because for any $0 < s < t$, we have: 1) The expression is measurable, and secondly:

$$\begin{aligned} \mathbb{E}(W^2(t)|F(s)) &= \mathbb{E}[|W(t) - W(s)|^2|\mathcal{F}(s)] + 2\mathbb{E}[W(t)W(s)|\mathcal{F}(s) \\ &\quad -\mathbb{E}[W^2(s)|\mathcal{F}(s)] \\ &= \mathbb{E}[|W(t) - W(s)|^2] + 2W(s)\mathbb{E}[W(t)|\mathcal{F}(s) - W^2(s) \\ &= t - s + 2W^2(s) - W^2(s) = t - s + W^2(s) \end{aligned}$$

Since $W(t) - W(s)$ is independent of $\mathcal{F}(s)$, has the normal distribution with mean 0 and variance $t - s$. $W(s)$ is $\mathcal{F}(s)$-measurable, and $W(t)$ is a martingale. It follows that $\mathbb{E}(W^2(t) - t|F(s)) = W^(s) - s$, so it is a martingale as well. This means, however, that the requested expression is **not** a martingale.

3. Let (X, Y) have a joint density function, given by

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \times$$
$$\exp\left\{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - \frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]\right\}$$

$\sigma_1, \sigma_2 > 0, |\rho| < 1, \mu_1, \mu_2 \in \mathbb{R}$.

Define

$$W = Y - \frac{\rho\sigma_2}{\sigma_1}X.$$

Answers:

Show that X and W are independent. It is sufficient to show that the covariance of X and W equals zero.

$$\begin{aligned}\mathbb{E}[(X-\mathbb{E}[X])(W-\mathbb{E}[W])] &= \mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])] - \mathbb{E}\left[\frac{\rho\sigma_2}{\sigma_1}(X-\mathbb{E}[X])^2\right] \\ &= \mathrm{Cov(X,Y)} - \frac{\rho\sigma_2}{\sigma_1}\sigma_1^2 = 0.\end{aligned}$$

4. Let X be a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and assume X has a density function $f(x)$ that is positive for every $x \in \mathbb{R}$. Let h be a strictly increasing, differentiable function satisfying

$$\lim_{y\to-\infty} h(y) = -\infty, \quad \lim_{y\to\infty} h(y) = \infty.$$

and define the random variable $Y = h(X)$. Let $g(y)$ be an arbitrary nonnegative function satisfying $\int_{-\infty}^{\infty} g(y)dy = 1$. We want to change the probability measure so that $g(y)$ is the density function for the random variable Y. To do this, we define

$$Z = \frac{g(h(X))h'(X)}{f(X)}$$

Answers:

a. Show that Z is nonnegative and $\mathbb{E}Z = 1$. Proof. Clearly $Z \geq 0$. Furthermore, we have

$$\begin{aligned}\mathbb{E}[Z] &= \mathbb{E}\left[\frac{g(h(X))h'(X)}{f(X)}\right] \\ &= \int_{-\infty}^{\infty} \frac{g(h(x))h'(x)}{f(x)} f(x)\mathrm{d}x = \int_{-\infty}^{\infty} g(h(x))h'(x)\mathrm{d}x \\ &= \int_{-\infty}^{\infty} g(u)\mathrm{d}u = 1\end{aligned}$$

b. Now define $\tilde{\mathbb{P}}$, as follows:

$$\tilde{\mathbb{P}}(A) = \int_A Z \mathrm{d}\mathbb{P}, \quad \text{for all } A \in \mathcal{F}$$

Show that Y has density g under $\tilde{\mathbb{P}}$.

$$\begin{aligned} \tilde{\mathbb{P}}(Y \leq A) &= \int_{h(x) \leq A} \frac{g(h(X))h'(X)}{f(X)} \mathrm{d}\mathbb{P} \\ &= \int_{-\infty}^{h^{-1}(a)} \frac{g(h(x))h'(x)}{f(x)} f(x)\mathrm{d}x = \int_{-\infty}^{h^{-1}(a)} g(h(x))\mathrm{d}h(x) \end{aligned}$$

By the change of variable formula, the last equation above equals to $\int_{-\infty}^{a} g(u)\mathrm{d}u$. So Y has density g under $\tilde{\mathbb{P}}$.