1. For any integrable random variable $X$ and any event $B \in \mathcal{F}$ such that $\mathbb{P}(B) \neq 0$, the conditional expectation of $X$ given $B$ is defined by

$$E[X|B] = \frac{1}{\mathbb{P}(B)} \int_B X \, d\mathbb{P}.$$

a. Show that if $X(\omega) = 1_A(\omega) = \begin{cases} 1 & \omega \in A, \\ 0 & \omega \notin A, \end{cases}$

then $E[1_A|B] = \mathbb{P}(A|B)$, where

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

is the conditional probability of $A$ given $B$. Furthermore, show that $E[X|\Omega] = E[X]$.

b. If $X$ and $Y$ are random variables and $E[Y|X] = c$, then show that $X$ and $Y$ are uncorrelated. (Hint: It’s sufficient to show that $\text{Cov}(X, Y) = E[XY] - E[X]E[Y] = 0$.)

c. Three coins, 10 cents, 20 cents and 50 cents are tossed. The values of those coins that land with heads up are added to give us the total amount $Z$. What is the expected total amount $Z$ given that two coins have landed with heads up?

2. Determine whether the following random variables are a martingale with respect to filtration $\mathcal{F}_t$ and give derivations for your statement.

a. $Y(t) = e^{W(t)} - t$;

b. $Z(t) = |W(t)|^2 - t^2$.

Z.O.Z. Remaining questions on the other side.
3. Let \((X, Y)\) have a joint density function, given by

\[
f_{X,Y}(x, y) = \frac{1}{2\pi \sigma_1 \sigma_2 \sqrt{1 - \rho^2}} \times \exp \left\{ -\frac{1}{2(1 - \rho^2)} \left[ \left( \frac{x - \mu_1}{\sigma_1} \right)^2 - \frac{2\rho(x - \mu_1)(y - \mu_2)}{\sigma_1 \sigma_2} + \left( \frac{y - \mu_2}{\sigma_2} \right)^2 \right] \right\}
\]

\(\sigma_1, \sigma_2 > 0, |\rho| < 1, \mu_1, \mu_2 \in \mathbb{R}\).

Define

\[ W = Y - \frac{\rho \sigma_2}{\sigma_1} X. \]

Show that \(X\) and \(W\) are independent. (Hint: Also here, it is sufficient to show that \(\text{Cov}(X, W) = 0\).)

4. Let \(X\) be a random variable on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), and assume \(X\) has a density function \(f(x)\) that is positive for every \(x \in \mathbb{R}\). Let \(h\) be a strictly increasing, differentiable function satisfying

\[
\lim_{y \to -\infty} h(y) = -\infty, \quad \lim_{y \to \infty} h(y) = \infty.
\]

and define the random variable \(Y = h(X)\). Let \(g(y)\) be an arbitrary nonnegative function satisfying \(\int_{-\infty}^{\infty} g(y)\,dy = 1\). We want to change the probability measure so that \(g(y)\) is the density function for the random variable \(Y\). To do this, we define

\[
Z = \frac{g(h(X))h'(X)}{f(X)}
\]

a. Show that \(Z\) is nonnegative and \(\mathbb{E}Z = 1\).

b. Now define \(\tilde{\mathbb{P}}\), as follows:

\[
\tilde{\mathbb{P}}(A) = \int_A Z \, d\mathbb{P}, \quad \text{for all } A \in \mathcal{F}
\]

Show that \(Y\) has density \(g\) under \(\tilde{\mathbb{P}}\).

Please, make sure that your name is written down on each of the submitted solution sheets.