Utrecht University
Mathematical Institute

Mid-Term Exam for Introduction to Financial Mathematics, WISB373

Friday May 20th 2022, 9:00 - 11:00 **(2 hours examination)**

For each of the eight exercises 10 points can be obtained.

1. For any integrable random variable X and any event $B \in \mathcal{F}$ such that $\mathbb{P}(B) \neq 0$, the conditional expectation of X given B is defined by

$$E[X|B] = \frac{1}{\mathbb{P}(B)} \int_B X \mathrm{d}\mathbb{P}.$$

a. Show that if

$$X(\omega) = \mathbf{1}_A(\omega) = \begin{cases} 1 & \omega \in A, \\ 0 & \omega \notin A, \end{cases}$$

then $\mathbb{E}[\mathbf{1}_A|B] = \mathbb{P}(A|B)$, where

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

is the conditional probability of A given B.
Furthermore, show that $\mathbb{E}[X|\Omega] = \mathbb{E}[X]$.

b. If X and Y are random variables and $\mathbb{E}[Y|X] = c$, then show that X and Y are uncorrelated. (Hint: It's sufficient to show that $\mathrm{Cov(X, Y)} = \mathbb{E}[\mathrm{XY}] - \mathbb{E}[\mathrm{X}]\mathbb{E}[\mathrm{Y}] = 0$.)

c. Three coins, 10 cents, 20 cents and 50 cents are tossed. The values of those coins that land with heads up are added to give us the total amount Z. What is the expected total amount Z given that two coins have landed with heads up?

2. Determine whether the following random variables are a martingale with respect to filtration $\mathcal{F}_t$ and give derivations for your statement.

a. $Y(t) = \mathrm{e}^{W(t)-t}$;

b. $Z(t) = |W(t)|^2 - t^2$.

Z.O.Z. Remaining questions on the other side.

3. Let (X,Y) have a joint density function, given by

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \times$$
$$\exp\left\{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - \frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]\right\}$$

$\sigma_1, \sigma_2 > 0, |\rho| < 1, \mu_1, \mu_2 \in \mathbb{R}$.

Define

$$W = Y - \frac{\rho\sigma_2}{\sigma_1}X.$$

Show that X and W are independent. (Hint: Also here, it is sufficient to show that $\mathrm{Cov(X,W)} = 0$.)

4. Let X be a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and assume X has a density function $f(x)$ that is positive for every $x \in \mathbb{R}$. Let h be a strictly increasing, differentiable function satisfying

$$\lim_{y\to-\infty} h(y) = -\infty, \quad \lim_{y\to\infty} h(y) = \infty.$$

and define the random variable $Y = h(X)$. Let $g(y)$ be an arbitrary nonnegative function satisfying $\int_{-\infty}^{\infty} g(y)dy = 1$. We want to change the probability measure so that $g(y)$ is the density function for the random variable Y. To do this, we define

$$Z = \frac{g(h(X))h'(X)}{f(X)}$$

a. Show that Z is nonnegative and $\mathbb{E}Z = 1$.

b. Now define $\tilde{\mathbb{P}}$, as follows:

$$\tilde{\mathbb{P}}(A) = \int_A Z d\mathbb{P}, \quad \text{for all } A \in \mathcal{F}$$

Show that Y has density g under $\tilde{\mathbb{P}}$.

Please, make sure that your name is written down on each of the submitted solution sheets.