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Stochastic Processes: Final, 2021-22
You are allowed Two two-sided A4 with any information that you desire

(1) Let $(X_n)_{n\geq 0}$ be a Markov chain with state space $I = \{1, 2, \cdots, N\}$ ($N > 1$) and transition probabilities given by

$$p_{ij} = \mathbb{P}(X_n = j | X_{n-1} = i) = \frac{|i-j|}{\sum_{k=1}^{N} |i-k|}.$$

(a) Prove that $(X_n)_{n\geq 0}$ is irreducible. (0.5 pt)

(b) Find the stationary distribution π and prove that if the initial distribution is π then $(X_n)_{n\geq 0}$ is time reversible. (1 pt)

(c) Find the asymptotic frequency that the Markov chain is in state i, $i = 1, \cdots, N$. (0.5 pt)

(d) Prove that $(X_n)_{n\geq 0}$ is aperiodic. (1 pt).

(2) A subway station has three lines, red, green and orange. Subways on each line arrive at the station according to three independent Poisson processes with rates 1 red subway every 10 minutes, 1 green subway every 15 minutes and 1 orange subway every 20 minutes.

(a) When you arrive at the station, what is the probability that the first subway that arrives is the green line? (1 pt)

(b) What is the expected time you have to wait before some subway arrive? (1 pt)

(c) You have been waiting for 20 minutes for the red line subway and have seen three orange line subways go by. What is the expected additional time you will have to wait for the red subway? (1 pt)

(3) Suppose that travellers arrive at a train station according to a Poisson process at a rate λ. Suppose that the train departs a time t, find the total expected waiting time of all the travellers that arrive in the interval $[0, t]$. (2 pt)

(4) Let $(X_t)_{t\geq 0}$ be a Poisson process with rate $\lambda > 0$ describing the occurrence of a certain phenomena. Suppose that occurrences independently come in two types, A and B. Type A occurs with probability p and B with probability $1-p$. Let Y_t be the number of occurrences of type A in the interval $[0, t]$, and Z_t the number of occurrences of type B in the interval $[0, t]$.

(a) Let $s < t$ and $0 \leq m \leq n$. Prove that $\mathbb{P}(X_s = m \,|\, X_t = n) = \binom{n}{m}(\frac{s}{t})^m(1-\frac{s}{t})^{n-m}$. (1 pt)

(b) Find $P(X_t = n | Y_t = m)$ for $m \leq n$. (1 pt)