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Stochastic Processes: Retake, 2021-22
You are allowed three two-sided A4 with any information that you desire

(1) Consider the Markov chain $(X_n)_{n\geq 0}$ with state space $I = \{1,2,3,4\}$ and transition matrix

$$\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{1}{4} & 0 & 0 & \frac{3}{4} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

(a) Determine the communicating classes, which ones are recurrent and which ones are transient. Justify your answer. (0.5 pt)

(b) For $i = 1,2,3,4$, determine the distribution of T_i under the conditional measure $\mathbb{P}_i$, where $T_i = \inf\{n \geq 1 : X_n = i\}$ is the first passage time to state i. (Hint: $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, for $|x| < 1$.) (1 pt)

(c) Determine $\mathbb{E}_i[T_i]$ for $i = 1,2,3,4$. (Hint: $\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}$, for $|x| < 1$.) (0.5 pt)

(2) Consider a Markov chain $(X_n)_{n\geq 0}$ with state space $I = \{0,1,2,\dots\}$ and transition probabilities

$$p_{i,i+1} = \frac{3}{4},\ p_{i,0} = \frac{1}{4},\ i \geq 0.$$

(a) Prove that the Markov chain is irreducible, positive recurrent and aperiodic. (1 pt)

(b) Find the stationary distribution π and determine $\lim_{n\to\infty} \mathbb{P}(X_n = j)$, $j \geq 0$. (1.5 pt)

(c) Assume the initial distribution is π, the stationary distribution, is the Markov chain time reversible? Justify your answer. (0.5 pt)

(d) Suppose we change the transition probabilities to

$$p_{i,i+1} = \frac{i+1}{i+2},\ p_{i,0} = \frac{1}{i+2},\ i \geq 0.$$

Prove that in this case, the Markov chain is still irreducible but is null-recurrent. (1 pt)

(3) Assume that passengers arrive at a train station according to a Poisson process with rate $\lambda = 2$ per minute. Suppose that the passengers independently come in two types: children and adults. The probability of a child arriving is $\frac{1}{3}$ and the probability of an adult arriving is $\frac{2}{3}$. Assume further that a train arrives every 10 minutes, so the first departing train is at $t = 10$.

(a) Find the expected arrival time of the tenth passenger, and the probability that the tenth passenger arrives at least two minutes after the ninth passenger. (0.5 pt)

(b) Find the probability that two passengers arrive in the time interval $[1,4)$ and three passengers arrive in $[3,5)$. (1.5 pts)

(c) Suppose 10 passengers arrived in the interval $[0,4]$, what is the probability that 4 of those are children? (0.5 pt)

(d) Find the total expected waiting time of all the passengers boarding the first train. (1.5 pts)