

**Stochastic Processes: Mid-Term, 2021-22**

- (1) Consider the Markov chain  $(X_n)_{n \geq 0}$  with state space  $I = \{1, 2, 3, 4\}$  and transition matrix

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} \\ \frac{3}{4} & 0 & \frac{1}{4} & 0 \end{bmatrix}.$$

- (a) Determine the communicating classes, which ones are closed? (1 pt)
- (b) Which classes are recurrent and which classes are transient? Justify your answer. (1 pt)
- (c) Determine  $\mathbb{E}_i[T_i]$  for  $i = 1, 2, 3, 4$ , where  $T_i = \inf\{n \geq 1 : X_n = i\}$  is the first passage time to state  $i$ . (2 pts)
- (2) Let  $P$  be the transition matrix of an irreducible Markov chain with state space  $\{1, 2, \dots, N\}$ . Define  $Q = \frac{1}{2}(I + P)$ , where  $I$  is the  $N \times N$  identity matrix.
- (a) Show that  $Q$  is a stochastic matrix which is irreducible and aperiodic. (2 pts)
- (b) Show that  $P$  and  $Q$  have the same stationary distributions, i.e if  $\pi$  is a probability vector then,  $\pi P = \pi$  if and only if  $\pi Q = \pi$ . (1 pt)
- (3) Consider three urns, one coloured red, one coloured white and one blue. The red urn contains 1 red and 4 blue balls; the white urn contains 3 white balls, 2 red balls and 2 blue balls; the blue urn contains 4 white, 3 red and 2 blue balls. At time 0 select a ball from the red urn, note its colour and return it back to the red urn. At time 1 choose a ball from the urn of the same colour as the previously selected ball (in this case at time 0), note its colour and return it to the urn it was selected from. Repeat this process at every time unit. Let  $X_n$  be the colour of the ball selected at time  $n$ . To simplify the notation, we will denote red colour by 0, a white colour by 1 and a blue colour by 2.
- (a) Argue why  $(X_n)_{n \geq 0}$  is a Markov chain. Determine its transition matrix  $P$  and initial distribution  $\lambda$ . (1.5 pts)
- (b) What is the long time probability of selecting a red, a white or a blue ball? (1.5 pts)