<u>Final exam, Mathematical Modelling</u> (WISB357)

Monday, 31 Jan 2022, 17:00-20:00, Educatorium Beta

- You may use your book and notes while working the problems.
- Write your name on each page you turn in, and additionally, on the first page, write your student number and the *total number of pages submitted.*
- For each question, motivation your answer.
- You may make use of results from previous subproblems, even if you have been unable to prove them.
- The maximum number of points per subproblem are given in italics between square brackets.
- Your grade is the total earned points divided by 3.
- The final exam weighs 40% in your grade for the course.

Integrity statement for online exams:
I hereby declare that I have prepared the solutions to this exam by myself, with no help from others, nor consulting any sources other than the course material, book and my own notes.
Name, student number, signature, date:

<u>Problem 1</u>. [*Reaction equations; stability; model interpretation.*] The following model is proposed for the spreading of disease in a heterogeneous population with two groups of susceptible individuals, those S_1 who recover after a mild infection, and those S_2 who die from the disease.

$$S_1 + I \xrightarrow{\beta_1} 2I$$
$$S_2 + I \xrightarrow{\beta_2} I$$
$$I \xrightarrow{\rho} R$$
$$R \xrightarrow{m} S_1$$

The infectious individuals are denoted by I and the recovered individuals by R. You may assume all rate parameters β_1, β_2, ρ, and m are positive, and we are only interested in nonnegative populations $S_1(t)$, $S_2(t)$, $I(t)$ and $R(t)$. Assume $S_1(0) > 0$, $S_2(0) > 0$, $I(0) > 0$, $R(0) = 0$.

(a) [2pts] Write down the system of ordinary differential equations corresponding to the above reaction formulas.

(b) [2pts] Find a linear conservation law and use it to eliminate the variable $R(t)$. By analyzing the stoichiometric matrix, prove that there are no other linear conservation laws.

(c) [2pts] By analyzing the equilibria and their stability, show that this model predicts that eventually the whole population of S_2 susceptibles will die from the disease. (You may assume that $\rho/\beta_1 < S_1(0) + I(0)$. You may also ignore any zero eigenvalues when determining stability.)

A vaccination program is introduced. An additional reaction is added to the model to reflect the principle that the vaccinated experience a milder form of the disease ($k \geq 0$):

$$S_2 \xrightarrow{k} R.$$

(d) [3pts] Find the equilibria and discuss their stability.

(e) [1pt] The parameter β_2 can be decreased by implementing social distancing. The parameter k can be influenced using an advertisement campaign for the vaccination program. Explain which measure is most effective if one wants to minimize the total number of deaths.

Problem 2. [*Singular perturbations.*] Consider the following two-point boundary value problem

$$\varepsilon^3 \frac{d^2y}{dx^2} + \left(\frac{\varepsilon}{1-\varepsilon x}\right)\frac{dy}{dx} + y^3 = x, \qquad y(0) = y(1) = 1, \quad 0 < \varepsilon < 1.$$

Construct an approximate solution as follows:

(a) [2pt] Derive the one-term outer expansion and show this can satisfy one of the boundary conditions.

(b) [5pt] Derive the one-term inner expansion and show this can satisfy the other boundary condition.

(c) [2pt] Use the matching condition to determine any free constants.

(d) [1pt] Give the composite solution.

Problem 3. [*Continuum mechanics; separation of variables.*] To a good approximation, the air in a long, narrow musical instrument of length ℓ satisfies the momentum equation from continuum mechanics in material coordinates:

$$R_0(A)\frac{\partial^2 U}{\partial t^2}(A,t) = R_0(A)F(A,t) + \frac{\partial T}{\partial A}(A,t), \qquad 0 < A < \ell, \quad t > 0,$$

where $R_0(A)$ is the density at time $t = 0$, $U(A,t)$ is the displacement of a cross-section of air, $F(A,t)$ is the net external body force, and $T(A,t)$ is the stress. Suppose that density is uniform (R_0 is independent of A) and the external body force is negligible $F(A,t) = 0$. The stress $T(A,t)$ is a linear function of the Lagrangian strain $\epsilon = \partial U/\partial A$ with Young's modulus $E > 0$. Furthermore, assume the instrument is closed at one end and open at the other such that the boundary conditions are

$$T(A,t)\big|_{A=0} = 0, \qquad U(A,t)\big|_{A=\ell} = 0.$$

(a) [2pts] Show that the momentum equation for the air in the instrument can be modelled by

$$\begin{aligned}
&\frac{\partial^2 U}{\partial t^2} = c^2 \frac{\partial^2 U}{\partial A^2}, \\
&\frac{\partial U}{\partial A}(0,t) = 0, \\
&U(\ell,t) = 0.
\end{aligned}$$

(b) [4pts] Find the eigenvalues λ_k and *orthonormal* eigenfunctions $X_k(A)$, $k = 0, \pm 1, \pm 2, \ldots$ of the eigenvalue problem

$$\begin{aligned} &X''(A) = \lambda X(A), \\ &X'(0) = 0, \\ &X(\ell) = 0. \end{aligned}$$

Explain why the associated eigenfunctions form a maximal orthonormal set.

(c) [4pts] Suppose the initial conditions satisfy

$$\begin{aligned} U(A,0) = f(A) &= \sum_{k=-\infty}^{\infty} f_k X_k(A), \\ \frac{\partial U}{\partial t}(A,0) = g(A) &= \sum_{k=-\infty}^{\infty} g_k X_k(A), \end{aligned}$$

Using the method of separation of variables, express the Fourier series solution of the momentum equation as given in part (a) in terms of the eigenfunctions $X_k(A)$.