

## Final exam, Mathematical Modelling (WISB357)

Monday, 31 Jan 2022, 17:00-20:00, Auditorium Beta

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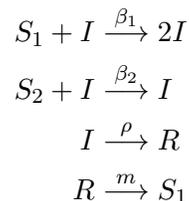
- You may use your book and notes while working the problems.
  - Write your name on each page you turn in, and additionally, on the first page, write your student number and the *total number of pages submitted*.
  - For each question, motivation your answer.
  - You may make use of results from previous subproblems, even if you have been unable to prove them.
  - The maximum number of points per subproblem are given in italics between square brackets.
  - Your grade is the total earned points divided by 3.
  - The final exam weighs 40% in your grade for the course.
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### Integrity statement for online exams:

*I hereby declare that I have prepared the solutions to this exam by myself, with no help from others, nor consulting any sources other than the course material, book and my own notes.*

Name, student number, signature, date:

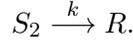
**Problem 1.** [*Reaction equations; stability; model interpretation.*] The following model is proposed for the spreading of disease in a heterogeneous population with two groups of susceptible individuals, those  $S_1$  who recover after a mild infection, and those  $S_2$  who die from the disease.



The infectious individuals are denoted by  $I$  and the recovered individuals by  $R$ . You may assume all rate parameters  $\beta_1$ ,  $\beta_2$ ,  $\rho$ , and  $m$  are positive, and we are only interested in nonnegative populations  $S_1(t)$ ,  $S_2(t)$ ,  $I(t)$  and  $R(t)$ . Assume  $S_1(0) > 0$ ,  $S_2(0) > 0$ ,  $I(0) > 0$ ,  $R(0) = 0$ .

- [2pts] Write down the system of ordinary differential equations corresponding to the above reaction formulas.
- [2pts] Find a linear conservation law and use it to eliminate the variable  $R(t)$ . By analyzing the stoichiometric matrix, prove that there are no other linear conservation laws.
- [2pts] By analyzing the equilibria and their stability, show that this model predicts that eventually the whole population of  $S_2$  susceptibles will die from the disease. (You may assume that  $\rho/\beta_1 < S_1(0) + I(0)$ . You may also ignore any zero eigenvalues when determining stability.)

A vaccination program is introduced. An additional reaction is added to the model to reflect the principle that the vaccinated experience a milder form of the disease ( $k \geq 0$ ):



- (d) [3pts] Find the equilibria and discuss their stability.
- (e) [1pt] The parameter  $\beta_2$  can be decreased by implementing social distancing. The parameter  $k$  can be influenced using an advertisement campaign for the vaccination program. Explain which measure is most effective if one wants to minimize the total number of deaths.

**Problem 2.** [*Singular perturbations.*] Consider the following two-point boundary value problem

$$\varepsilon^3 \frac{d^2 y}{dx^2} + \left( \frac{\varepsilon}{1 - \varepsilon x} \right) \frac{dy}{dx} + y^3 = x, \quad y(0) = y(1) = 1, \quad 0 < \varepsilon < 1.$$

Construct an approximate solution as follows:

- (a) [2pt] Derive the one-term outer expansion and show this can satisfy one of the boundary conditions.
- (b) [5pt] Derive the one-term inner expansion and show this can satisfy the other boundary condition.
- (c) [2pt] Use the matching condition to determine any free constants.
- (d) [1pt] Give the composite solution.

**Problem 3.** [*Continuum mechanics; separation of variables.*] To a good approximation, the air in a long, narrow musical instrument of length  $\ell$  satisfies the momentum equation from continuum mechanics in material coordinates:

$$R_0(A) \frac{\partial^2 U}{\partial t^2}(A, t) = R_0(A) F(A, t) + \frac{\partial T}{\partial A}(A, t), \quad 0 < A < \ell, \quad t > 0,$$

where  $R_0(A)$  is the density at time  $t = 0$ ,  $U(A, t)$  is the displacement of a cross-section of air,  $F(A, t)$  is the net external body force, and  $T(A, t)$  is the stress. Suppose that density is uniform ( $R_0$  is independent of  $A$ ) and the external body force is negligible  $F(A, t) = 0$ . The stress  $T(A, t)$  is a linear function of the Lagrangian strain  $\epsilon = \partial U / \partial A$  with Young's modulus  $E > 0$ . Furthermore, assume the instrument is closed at one end and open at the other such that the boundary conditions are

$$T(A, t)|_{A=0} = 0, \quad U(A, t)|_{A=\ell} = 0.$$

- (a) [2pts] Show that the momentum equation for the air in the instrument can be modelled by

$$\begin{aligned} \frac{\partial^2 U}{\partial t^2} &= c^2 \frac{\partial^2 U}{\partial A^2}, \\ \frac{\partial U}{\partial A}(0, t) &= 0, \\ U(\ell, t) &= 0. \end{aligned}$$

- (b) [4pts] Find the eigenvalues  $\lambda_k$  and *orthonormal* eigenfunctions  $X_k(A)$ ,  $k = 0, \pm 1, \pm 2, \dots$  of the eigenvalue problem

$$\begin{aligned}X''(A) &= \lambda X(A), \\X'(0) &= 0, \\X(\ell) &= 0.\end{aligned}$$

Explain why the associated eigenfunctions form a maximal orthonormal set.

- (c) [4pts] Suppose the initial conditions satisfy

$$\begin{aligned}U(A, 0) = f(A) &= \sum_{k=-\infty}^{\infty} f_k X_k(A), \\ \frac{\partial U}{\partial t}(A, 0) = g(A) &= \sum_{k=-\infty}^{\infty} g_k X_k(A),\end{aligned}$$

Using the method of separation of variables, express the Fourier series solution of the momentum equation as given in part (a) in terms of the eigenfunctions  $X_k(A)$ .