Problem 1. [Reaction equations; stability; model interpretation.] The following model is proposed for the spreading of disease in a heterogeneous population with two groups of susceptible individuals, those $S_1$ who recover after a mild infection, and those $S_2$ who die from the disease.

$$
S_1 + I \xrightarrow{\beta_1} 2I \\
S_2 + I \xrightarrow{\beta_2} I \\
I \xrightarrow{\rho} R \\
R \xrightarrow{m} S_1
$$

The infectious individuals are denoted by $I$ and the recovered individuals by $R$. You may assume all rate parameters $\beta_1$, $\beta_2$, $\rho$, and $m$ are positive, and we are only interested in nonnegative populations $S_1(t)$, $S_2(t)$, $I(t)$ and $R(t)$. Assume $S_1(0) > 0$, $S_2(0) > 0$, $I(0) > 0$, $R(0) = 0$.

(a) [2pts] Write down the system of ordinary differential equations corresponding to the above reaction formulas.

(b) [2pts] Find a linear conservation law and use it to eliminate the variable $R(t)$. By analyzing the stoichiometric matrix, prove that there are no other linear conservation laws.

(c) [2pts] By analyzing the equilibria and their stability, show that this model predicts that eventually the whole population of $S_2$ susceptibles will die from the disease. (You may assume that $\rho/\beta_1 < S_1(0) + I(0)$. You may also ignore any zero eigenvalues when determining stability.)
A vaccination program is introduced. An additional reaction is added to the model to reflect the principle that the vaccinated experience a milder form of the disease ($k \geq 0$):

$$S_2 \xrightarrow{k} R.$$ 

(d) [3pts] Find the equilibria and discuss their stability.

(e) [1pt] The parameter $\beta_2$ can be decreased by implementing social distancing. The parameter $k$ can be influenced using an advertisement campaign for the vaccination program. Explain which measure is most effective if one wants to minimize the total number of deaths.

Problem 2. [Singular perturbations.] Consider the following two-point boundary value problem

$$\varepsilon^3 \frac{d^2 y}{dx^2} + \left( \frac{\varepsilon}{1 - \varepsilon x} \right) \frac{dy}{dx} + y^3 = x, \quad y(0) = y(1) = 1, \quad 0 < \varepsilon < 1.$$ 

Construct an approximate solution as follows:

(a) [2pt] Derive the one-term outer expansion and show this can satisfy one of the boundary conditions.

(b) [5pt] Derive the one-term inner expansion and show this can satisfy the other boundary condition.

(c) [2pt] Use the matching condition to determine any free constants.

(d) [1pt] Give the composite solution.

Problem 3. [Continuum mechanics; separation of variables.] To a good approximation, the air in a long, narrow musical instrument of length $\ell$ satisfies the momentum equation from continuum mechanics in material coordinates:

$$R_0(A) \frac{\partial^2 U}{\partial t^2}(A, t) = R_0(A) F(A, t) + \frac{\partial T}{\partial A}(A, t), \quad 0 < A < \ell, \quad t > 0,$$

where $R_0(A)$ is the density at time $t = 0$, $U(A, t)$ is the displacement of a cross-section of air, $F(A, t)$ is the net external body force, and $T(A, t)$ is the stress. Suppose that density is uniform ($R_0$ is independent of $A$) and the external body force is negligible $F(A, t) = 0$. The stress $T(A, t)$ is a linear function of the Lagrangian strain $\epsilon = \frac{\partial U}{\partial A}$ with Young’s modulus $E > 0$. Furthermore, assume the instrument is closed at one end and open at the other such that the boundary conditions are

$$T(A, t)\big|_{A=0} = 0, \quad U(A, t)\big|_{A=\ell} = 0.$$ 

(a) [2pts] Show that the momentum equation for the air in the instrument can be modelled by

$$\frac{\partial^2 U}{\partial t^2} = \frac{c^2}{\ell^2} \frac{\partial^2 U}{\partial A^2}, \quad \frac{\partial U}{\partial A}(0, t) = 0, \quad U(\ell, t) = 0.$$
(b) [4pts] Find the eigenvalues $\lambda_k$ and orthonormal eigenfunctions $X_k(A), k = 0, \pm 1, \pm 2, \ldots$ of the eigenvalue problem

\[
\begin{align*}
X''(A) &= \lambda X(A), \\
X'(0) &= 0, \\
X(\ell) &= 0.
\end{align*}
\]

Explain why the associated eigenfunctions form a maximal orthonormal set.

(c) [4pts] Suppose the initial conditions satisfy

\[
\begin{align*}
U(A, 0) &= f(A) = \sum_{k=-\infty}^{\infty} f_k X_k(A), \\
\frac{\partial U}{\partial t}(A, 0) &= g(A) = \sum_{k=-\infty}^{\infty} g_k X_k(A),
\end{align*}
\]

Using the method of separation of variables, express the Fourier series solution of the momentum equation as given in part (a) in terms of the eigenfunctions $X_k(A)$. 