Exam Manifolds (November 8, 2021)

Note: please do not forget to write down your name and student number. Also, please motivate your answers. The first exercise is worth 1.5 point, while in the second exercise all the questions (a)-(p) are worth 0.5p except for the ones in blue which are worth 1p. The mark for the exam is the minimum between 10 and the total number of points you collect.

Exercise 1. (*1.5 pt*) Show that, on any manifold M,

$$L_{f\cdot X}(\omega) = f \cdot L_X(\omega) + df \wedge i_X(\omega)$$

for any smooth function $f \in C^\infty(M)$, any vector field $X \in \mathfrak{X}(M)$ and any differential form $\omega \in \Omega^k(M)$

Exercise 2. We consider the space of two by two matrices with real coefficients, identified to the Euclidean space $\mathbb{R}^4$ with coordinate functions denoted by x, y, z, t:

$$N := \left\{ A = \begin{pmatrix} x & y \\ z & t \end{pmatrix} : x, y, z, t \in \mathbb{R} \right\}$$

and, inside it, the space of matrices of determinant 1:

$$M := \left\{ A = \begin{pmatrix} x & y \\ z & t \end{pmatrix} : xt - yz = 1 \right\}.$$

First show that:

(a) M is an embedded submanifold of N.

(b) The following define vector fields tangent to M:

$$V^1 := z\frac{\partial}{\partial x} + t \cdot \frac{\partial}{\partial y}$$

$$V^2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - z\frac{\partial}{\partial z} - t\frac{\partial}{\partial t}.$$

$$V^3 = -y \cdot \frac{\partial}{\partial x} + x \cdot \frac{\partial}{\partial y} - t\frac{\partial}{\partial z} + z\frac{\partial}{\partial t}$$

(c) $f : M \to \mathbb{R}^2, \quad f\begin{pmatrix} x & y \\ z & t \end{pmatrix} = (z, t)$ is a submersion.

(d) Because t is used to denote the last coordinate in M, the time-parameter of curves will be denoted by s. Compute the flows of the vector fields V_1, V_2 and V_3 and show they are of type

$$\phi^s_{V_1}(A) = A_1(s) \cdot A, \quad \phi^s_{V_2}(A) = A_2(s) \cdot A, \quad \phi^s_{V_3}(A) = A \cdot A_3(s)$$

where $A_i(s)$ are matrices that depend on s. For instance, you should get

$$\phi^s_{V_1} = \begin{pmatrix} x + s\cdot z & y + s\cdot t \\ z & t \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x & y \\ z & t \end{pmatrix}, \quad \text{hence } A_1(s) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

(e) Show that $[V_1, V_3] = 0$ and $[V_2, V_3] = 0$.

(f) Show that $[V_1, V_2] = 2V_1$.

(g) Find another vector field V on M which admits the following integral curve:

$$\gamma(s) = A_1(s) \cdot A_3(s) = \begin{pmatrix} \cos s - s\cdot \sin s & \sin s + s \cdot \cos s \\ -\sin s & \cos s \end{pmatrix}$$

(h) Given an example of a 1-form θ on N which is nonzero but such that $\theta|_M = 0$.

(i) Using the 1-forms on M:

$$\theta_1 = -y\cdot dx + x \cdot dy + \frac{x^2+y^2}{z^2+t^2}\left(t\cdot dz - z\cdot dt\right),$$

$$\theta_2 = -\frac{1}{z^2+t^2}(z\cdot dz + t\cdot dt), \quad \theta_3 = \frac{1}{z^2+t^2}(-t\cdot dz + z \cdot dt),$$

show that V^1, V^2, V^3 induce a basis of T_pM at each $p \in M$.

(j) Show that $d\theta_2 = 0$ and $d\theta_3 = 0$.

(k) Consider now $\omega := dx \wedge dy \wedge dz$ as a form on M (the restriction to M). Compute $i_{V^1}(\omega)$, and then $\omega(V^1, V^2, V^3)$.

(l) Show that ω is not a volume form on M and find all $p \in M$ at which $\omega_p = 0$.

(m) Show that $L_{V^1}(\omega) = -dx \wedge dz \wedge dt$ in two ways: one using Cartan's formula, and one using flows.

(n) Consider the following parametrization of M:

$$\begin{cases} x = r\cdot \cos\alpha - \frac{u}{r}\sin\alpha & y = r\cdot\sin\alpha + \frac{u}{r}\cos\alpha \\ z = -\frac{1}{r}\sin\alpha & t = \frac{1}{r}\cos\alpha \end{cases} \qquad u \in \mathbb{R},\ r \in (0,\infty),\ \alpha \in (0, 2\pi)$$

which we interpret as the inverse χ^{-1} of a chart of M. What is the domain U of χ?

(o) Hence the u, r, α determined by the equations above (all functions of (x, y, z)) are precisely the components $\chi_1(x,y,z)$, $\chi_2(x,y,z)$ and $\chi_3(x,y,z)$ of $\chi(x,y,z)$. Show that the resulting vector field $\frac{\partial}{\partial \chi_1}$ coincides with V_1 at all points $p \in U$.

(p) Let $\mu := \theta_1 \wedge \theta_1 \wedge \theta_2$. Compute the coefficient of μ w.r.t. the chart χ,

$$f^\mu_\chi \in C^\infty(\mathbb{R} \times (0,\infty) \times (0, 2\pi)).$$