Exam Manifolds (RETAKE, December 2021)

Note: Please give all details. All the questions below are worth 0.5 points except for the ones in blue which are worth 1 pt (with a grand total of 11.5pt, of which only 10 are needed to receive the maximum mark).

Exercise 1. On the 2-sphere S^2, on which we use the coordinate functions $(x, y, z) \in \mathbb{R}^3$:

(a) Show that the function

$$f : S^2 \to S^2, \quad f(x, y, z) = (x^2 + y^2 - z^2, 2yz, 2xz)$$

is not a submersion.

(b) Show that $(xz \cdot dy - yz \cdot dx) \wedge dz = (1 - z^2) \cdot dx \wedge dy$ (an equality of 2-forms on S^2).

(c) Show that the pull-back via f of the volume form $\sigma = x \cdot dy \wedge dz + y \cdot dz \wedge dx + z \cdot dx \wedge dy$ satisfies

$$f^*\sigma = 4dx \wedge dy = 4z \cdot \sigma.$$

(d) Find the points $p \in S^2$ at which f fails to be a submersion.

(e) Compute $\int_{S^2} f^*\sigma$ (where you can use any orientation on S^2 that you want).

Exercise 2. Let $F : M \to N$ be a smooth map between two manifolds. Recall that, for vector fields $X \in \mathfrak{X}(M)$, $V \in \mathfrak{X}(N)$, one says that X is F-projectable to V if

$$(dF)_p(X_p) = V_{F(p)}$$

for all $p \in M$. Show that:

(a) If $X \in \mathfrak{X}(M)$ if F-projectable to $V \in \mathfrak{X}(N)$, then $L_X(F^*(f)) = F^*(L_V(f))$ for all $f \in C^\infty(N)$, where $F^* : C^\infty(N) \to C^\infty(M)$, $F^*(f) = f \circ F$.

(b) Then show that the converse of (a) holds as well.

(c) Deduce that, if $X \in \mathfrak{X}(M)$ is F-projectable to $V \in \mathfrak{X}(N)$ and $Y \in \mathfrak{X}(M)$ is F-projectable to $W \in \mathfrak{X}(N)$ then $[X, Y]$ is F-projectable to $[V, W]$.

Exercise 3. Assume that M is a compact manifold, $f : M \to S^1$ is a smooth map and that there exists a vector field $V \in \mathfrak{X}(M)$ that is f-projectable to $\frac{\partial}{\partial \varphi} = -y \cdot \frac{\partial}{\partial x} + x \cdot \frac{\partial}{\partial y} \in \mathfrak{X}(S^1)$. We also consider the pull-back via f of the canonical 1-form $d\varphi = -y \cdot dx + x \cdot dy \in \Omega^1(S^1)$,

$$\omega := f^*(d\varphi) \in \Omega^1(M).$$

Show that:

(a) For each $\alpha \in \mathbb{R}$, the fiber

$$M_\alpha := \{p \in M : f(p) = e^{i\alpha}\}$$

is an embedded submanifold of M.

(b) ω is closed.

(c) $\omega(V) = 1$.

(d) $\mathcal{L}_V(\omega) = 0$.

(e) $\phi_V^t(p) \in M_{\alpha+t}$ for any $p \in M_\alpha$, and that the flow of V gives rise to diffeomorphisms

$$\phi_V^t|_{M_\alpha} : M_\alpha \xrightarrow{\sim} M_{\alpha+t} \quad \text{(for all } t, \alpha \in \mathbb{R}\text{)}.$$

Exercise 4. On $\mathbb{R}^3$ we define the following operation ("product"):

$$(x_1, y_1, z_1) \star (x_2, y_2, z_2) := (x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1 y_2)$$

and we consider the associated left translations, one for each $p \in \mathbb{R}^3$,

$$L_p : \mathbb{R}^3 \to \mathbb{R}^3, \quad L_p(q) := p \star q.$$

We define the vector fields $X^1, X^2, X^3 \in \mathfrak{X}(\mathbb{R}^3)$ which, at $p \in \mathbb{R}^3$, are obtained by applying the differential of L_p at the origin, $(dL_p)_0 : T_0\mathbb{R}^3 \to T_p\mathbb{R}^3$, to

$$\left(\frac{\partial}{\partial x}\right)_0, \quad \left(\frac{\partial}{\partial y}\right)_0, \quad \left(\frac{\partial}{\partial z}\right)_0.$$

(a) Compute X^1, X^2, X^3 explicitly and show that they are dual to

$$\theta_1 = dx, \quad \theta_2 = dy, \quad \theta_3 = -x dy + dz,$$

i.e. $\theta_i(X^j) = \delta_i^j$ (1 if $i = j$ and 0 otherwise).

(b) Show that $[X^1, X^2] = X^3, [X^2, X^3] = 0, [X^3, X^1] = 0$.

(c) Find a vector field V such that $\theta^1(V) = 1$, $\theta^2(V) = 0$ and which commutes with X^1, X^2, X^3 (i.e. $[V, X^1] = 0$, $[V, X^2] = 0$, $[V, X^3] = 0$).

(d) For V that you found in (c), compute the flow ϕ_V^t of V explicitly.

(e) Show that ϕ_V^t preserves θ_3, i.e. $(\phi_V^t)^*\theta_3 = \theta_3$.

(f) Deduce that $\mathcal{L}_V(\theta_3) = 0$.

(g) Prove directly (by an algebraic computation) that $\mathcal{L}_V(\theta_3) = 0$.