Final exam

Topologie en Meetkunde, Block 3, 2022

Instructions

- The exam is closed-book.
- Write your name and student number in all the pages of the exam.
- You may write your solutions in either Dutch or English.
- You must justify the claims you make.
- You may use results from the lectures, but you must provide a clear statement (with complete hypothesis and conclusion).
- Try to write with clear handwriting. Structure your explanations clearly, using one paragraph for each new idea and one sentence for each particular claim.
- Advice: Read all the exam in the beginning and address first the questions that you find easier.

Questions

Exercise 1 (1.5 points)**.** Find two inclusions $i, j : \mathbb{S}^1 \to \mathbb{S}^1 \vee \mathbb{S}^1$ such that:

- $i(\mathbb{S}^1)$ is a retract.
- $j(\mathbb{S}^1)$ is a retract.
- i and j are not homotopic.

Then, find two non-homotopic retractions $f, g : \mathbb{S}^1 \vee \mathbb{S}^1 \to i(\mathbb{S}^1)$.

Note: In order to define the wedge $\mathbb{S}^1 \vee \mathbb{S}^1$ you have to pick basepoints but, in this particular case, the resulting space does not depend on these choices (up to homeomorphism).

Exercise 2 (0.75 points)**.** Let (X, p) be a pointed topological space. Let $\psi : \pi_1(\mathbb{S}^1, 1) \to \pi_1(X, p)$ be a group homomorphism. Show that there is a map $f : (\mathbb{S}^1, 1) \to (X, p)$ such that $f_* = \psi$.

Exercise 3 (0.75 points)**.** Consider T^2 given by its standard planar representation with one vertex p, two edges a and b, and one (square) face D. Let A be its 1-skeleton and let $\iota : A \to T^2$ be the inclusion.

Let $f : A \to \mathbb{S}^1$ be a map. Prove that f can be extended to T^2; that is, there is $g : T^2 \to \mathbb{S}^1$ such that $f = g \circ \iota$.

Exercise 4 (1 point)**.** Prove that $[\mathbb{S}^2 \vee \mathbb{S}^1, \mathbb{S}^1] \cong \mathbb{Z}$ (as sets).

Hint: You may want to use $[\mathbb{S}^1, \mathbb{S}^1] \cong \mathbb{Z}$, as proven in class.

Exercise 5 (2 points)**.** The suspension of a space Z is defined as:

$$\Sigma Z := (Z \times [0, 1])/ \sim,$$

where $(z, 0) \sim (z', 0)$ and $(z, 1) \sim (z', 1)$ for every $z, z' \in Z$.

Let $(X, p) := \vee^k(\mathbb{S}^1, 1)$.

- Endow ΣX with a cell structure; be explicit about the number of cells used and their attaching maps.
- Prove that ΣX is simply-connected.

Hint: You can use the cell structure on X as a guide to produce the cell structure of ΣX. You may want to think about the case $k = 1$ first.

Exercise 6 (2 points)**.** Let A, B be two copies of the torus $T^2 := \mathbb{S}^1 \times \mathbb{S}^1$. Let $a \in \mathbb{Z}$. Define the space

$$C := (A \coprod B)/(A \ni (z, 0) \cong (z, z^a) \in B).$$

- Compute the fundamental group of C.
- Compute the first homology of C.
- Prove that C is not a surface.

Exercise 7 (2 points)**.** Let K be the Klein bottle. Fix a basepoint $p \in K$. Prove the following statements:

- All the covering spaces of K with finitely many sheets have Euler characteristic zero.
- There is a covering space of K that is orientable.
- There are two path-connected, 2-sheeted, pointed covering spaces of (K, p) that are not isomorphic (as elements in $\mathrm{Cover}(K, p)$).