Retake

Topologie en Meetkunde, Block 3, 2022

Instructions

- Write your name and student number in all the pages of the exam.
- You have three hours to answer the questions below. Those with extra time will be allotted an additional 30 minutes.
- The exam is closed book.
- You may write your solutions in either Dutch or English.
- You must justify the claims you make.
- You may use results from the lectures, but you must provide a clear statement (with complete hypothesis and conclusion).
- Try to write with clear handwriting. Structure your explanations clearly, using one paragraph for each new idea and one sentence for each particular claim.
- Advice: Read all the exam in the beginning and address first the questions that you find easier.

Questions

Exercise 1 (3 points)**.** Prove, or provide a counterexample to, the following statements:

- Let A and B be homotopy equivalent spaces. Then A is compact if and only if B is compact.
- For all positive integers n, there is a cell structure on $\mathbb{S}^1$ with n vertices.
- Let Σ be a closed, path-connected, non-orientable surface. Then, Σ is not simply-connected.
- There is a (pointed) surface (C, c) whose fundamental group is not abelian.

Exercise 2 (1 point)**.** Fix spaces A, B, and K. Suppose that $K \subset A$ is a deformation retract. Let $\pi : \widetilde{B} \to B$ be a covering space. Assume that all these spaces are path-connected and locally contractible.

Prove that the following statements are equivalent:

- $f : A \to B$ lifts to a map $\widetilde{f} : A \to \widetilde{B}$.
- $g := f|_K : K \to B$ lifts to a map $\widetilde{g} : K \to \widetilde{B}$.

Exercise 3 (2.5 points)**.** Let a be a positive integer. Consider a copies $\{S_i\}_{i=1,\cdots,a}$ of the 2-sphere, with north poles $\{n_i \in S_i\}_{i=1,\cdots,a}$ and south poles $\{s_i \in S_i\}_{i=1,\cdots,a}$. Define

$$X := \left(\coprod_{i=1,\cdots,a} S_i \right) / \{n_i \cong s_{i+1} \text{ for every } i < a \text{ and } n_a \cong s_1\}.$$

- Endow X with a CW-structure. Be explicit about the number of cells you use and their attaching maps.

- Is X a surface?
- Compute the fundamental group of X.

Exercise 4 (2.5 points)**.** Let X be as in the previous exercise. Given a positive integer b:

- Find a path-connected covering space $\pi : Y_b \to X$ with b sheets.
- Compute the fundamental group of Y_b.
- Fix a basepoint $p \in X$ and a lift $q \in Y_b$. Describe the map $\pi_* : \pi_1(Y_b, q) \to \pi_1(X, p)$ (by explaining what it does on generators).

Then, provide a complete list of all the pointed, path-connected covering spaces of (X, p), up to isomorphism.

Exercise 5 (1 point)**.** Let $(W, p) := (\mathbb{S}^1, 1) \vee (\mathbb{S}^1, 1)$. Fix a positive integer $k \geq 2$. Prove that there is a path-connected covering space of (W, p) with fundamental group isomorphic to $*_k\mathbb{Z}$. **Hint:** Find examples with $k = 2, 3, 4$ and try to see what the pattern is.