Student number: .... Name: ....

Midterm exam

Topologie en Meetkunde, Block 3, 2022

Instructions

- The exam is closed-book.
- Write your name and student number in all the pages of the exam.
- You may write your solutions in either Dutch or English.
- You must justify the claims you make.
- You may use results from the lectures, but you must provide a clear statement (with complete hypothesis and conclusion).
- Try to write with clear handwriting. Structure your explanations clearly, using one paragraph for each new idea and one sentence for each particular claim.
- Advice: Read all the exam in the beginning and address first the questions that you find easier.

Questions

Exercise 1 (1.5 points)**.** Let $A \subset X$ be a retract. Show that:

- If X is contractible, then A is also contractible.
- If A is contractible, X need not be contractible.

Exercise 2 (2 points)**.** Let X, Y and A be topological spaces. Let $f : X \to Y$ be a continuous map. Recall that $[Y, A]$ denotes the set of equivalence classes of continuous maps $Y \to A$ up to homotopy. We define the **pullback** of f to be:

$$\begin{aligned} f^* : [Y, A] &\longrightarrow [X, A], \\ [g] &\longrightarrow f^*([g]) := [g \circ f]. \end{aligned}$$

Show that:

- f^* is a well-defined function.
- Given homotopic maps $f_0, f_1 : X \to Y$, it follows that $f_0^* = f_1^*$.
- If f is a homotopy equivalence, then f^* is a bijection.

Exercise 3 (1.5 points)**.** Recall that the wedge product is defined as:

$$(A, a) \vee (B, b) = ((A \coprod B)/\{A \ni a \cong b \in B\}, [a] = [b])$$

Consider the space $(Y, p) := (\mathbb{S}^1, 1) \vee (\mathbb{D}^2, 1)$.

- Prove that Y deformation retracts to its first factor $\mathbb{S}^1$.
- Compute the fundamental group of (Y, p).

Exercise 4 (1 point)**.** Show that $\Pi_1(X)$ is contractible if and only if X is contractible. **Hint:** You may want to invoke Exercise 1.

Exercise 5 (1 point)**.** Let $\mathbb{S}^2 \subset \mathbb{R}^3$ be the sphere. Let $\mathbb{S}^1 \subset \mathbb{R}^2 \subset \mathbb{R}^3$ be its equator. Show that there is no retraction $r : \mathbb{S}^2 \to \mathbb{S}^1$.

Exercise 6 (1 point)**.** Let (Y, p) be as in the Exercise 3. Consider its pointed subspace $(X, p) := (\mathbb{S}^1, 1) \vee (\mathbb{S}^1, 1)$.

- Provide a formula for the homomorphism $i_* : \pi_1(X, p) \to \pi_1(Y, p)$ induced by the inclusion $i : X \to Y$.
- Compute the kernel of i_*.
- Show that there is no retraction $r : Y \to X$.

Note: You may use $\pi_1(X, p) \cong \mathbb{Z} * \mathbb{Z}$ without proof.

Exercise 7 (2 points)**.** Let (Y, p) be as in the Exercise 3. Provide a explicit description of its universal cover

$$\pi : (\tilde{Y}, \tilde{p}) \longrightarrow (Y, p).$$

Describe π and prove it is a covering map. Show that $\tilde{Y}$ is contractible.

Note: Y itself was assembled using a circle and a disc. Your description of $\tilde{Y}$ should also use elementary pieces and be equally explicit. **Hint:** You may recall from our last lecture that the covering spaces of a cell-complex are also cell-complexes.