**Exam WISB326, June 28, 2022, 17:00-20:00**
**Solutions and grading scheme**

Exam problems

- Let $k$ be an algebraically closed field of characteristic 0.

(1) (12 points) Let $X = V((x_1+x_2)(2x_1+x_3), (x_2-x_3^2)(4x_1^2-x_3^2)) \subseteq \mathbb{A}^3(k)$. Determine the irreducible components of $X$.
**Solution**
Since $4x_1^2 - x_3^2 = (2x_1 + x_3)(2x_1 - x_3)$, using the properties of $V(.)$ we have

$$\begin{aligned}X &= V(2x_1 + x_3) \cup V(x_1 + x_2, (x_2 - x_3^2)(2x_1 - x_3)) \\ &= V(2x_1 + x_3) \cup (V(x_1 + x_2) \cap V((x_2 - x_3^2)(2x_1 - x_3))) \\ &= V(2x_1 + x_3) \cup (V(x_1 + x_2) \cap V(x_2 - x_3^2)) \cup (V(x_1 + x_2) \cap V(2x_1 - x_3)) \\ &= V(2x_1 + x_3) \cup V(x_1 + x_2, x_2 - x_3^2) \cup V(x_1 + x_2, 2x_1 - x_3).\end{aligned}$$

Now, $2x_1 + x_3$ is an irreducible polynomial, $(x_1 + x_2, 2x_1 - x_3)$ is a prime ideal as it is generated by linear polynomials, and $k[x_1, x_2, x_3]/(x_1 + x_2, x_2 - x_3^2) \cong k[x_3]$ is an integral domain, hence $X_1 := V(2x_1 + x_3)$, $X_2 := V(x_1 + x_2, x_2 - x_3^2)$ and $X_3 := V(x_1 + x_2, 2x_1 - x_3)$ are irreducible. We observe that $X_1 \cap X_3 = \{(0, 0, 0)\}$ is a proper subset of both $X_1$ and $X_3$, hence $X_1 \not\subseteq X_3$ and $X_3 \not\subseteq X_1$. Similarly, $X_1 \not\subseteq X_2$ and $X_2 \not\subseteq X_1$ as $X_1 \cap X_2 = \{(0, 0, 0), (-\frac{1}{4}, \frac{1}{4}, \frac{1}{2})\}$ is a proper subset of both $X_1$ and $X_3$. Finally, $X_2 \cap X_3 = \{(0, 0, 0), (-\frac{1}{4}, \frac{1}{4}, -\frac{1}{2})\}$ is a proper subset of both $X_2$ and $X_3$, hence $X_2 \not\subseteq X_3$ and $X_3 \not\subseteq X_2$. Thus $X_1, X_2, X_3$ are the irreducible components of $X$.
6 points for computations
3 points for irreducibility of the components
3 points for components not contained in each other

(2) Let $X \subseteq \mathbb{A}^n(k)$ be an affine algebraic set.
(a) (4 points) Show that $X = V(I(X))$.
(b) (4 points) Show that $X$ is a point if and only if $I(X)$ is a maximal ideal.
(c) (6 points) Let $P = (a_0 : \cdots : a_n) \in \mathbb{P}^n(k)$. Show that the ideal $I(P) \subseteq k[x_0, \ldots, x_n]$ is generated by the polynomials $a_i x_j - a_j x_i$ for $i, j \in \{0, \ldots, n\}$.
**Solution**
(a) By definition $I(X)$ is the set of polynomials that vanish on $X$, and $V(I(X))$ is the set of common zeros of all polynomials in $I(X)$. Hence, $X \subseteq V(I(X))$. Since $X$ is an algebraic set, there is an ideal $J \subseteq k[x_1, \ldots, x_n]$ such that $X = V(J)$. By the Nullstellensatz $I(X) = \sqrt{J} \supseteq J$. Thus $V(I(X)) = V(\sqrt{J}) \subseteq V(J) = X$.

2 points for one inclusion
2 points for the reverse inclusion: 1 for $J \subseteq I(X)$ and 1 for conclusion.

(b) Assume that $X$ is a point, $X = (a_1, \ldots, a_n)$. Then $x_i - a_i \in I(X)$ for all $i \in \{1, \ldots, n\}$. We observe that $(x_1 - a_1, \ldots, x_n - a_n)$ is a maximal ideal as it is the kernel of the morphism $k[x_1, \ldots, x_n] \to k$ that sends $x_i$ to $a_i$ for all $i \in \{1, \ldots, n\}$. Since $X \neq \emptyset$, then $I(X)$ is a proper ideal. Hence, $(x_1 - a_1, \ldots, x_n - a_n) = I(X)$ by maximality of $(x_1 - a_1, \ldots, x_n - a_n)$. For the converse implication, let $x \in X$ be a point. If $X \neq \{x\}$, then $I(X) \subsetneq I(\{x\})$ by part (a). Since $I(\{x\})$ is a proper ideal (for example, because we just proved above that it is a maximal ideal), we conclude that $I(X)$ is not a maximal ideal.
2 points for one implication
2 points for the reverse implication

(c) Observe that $a_i x_j - a_j x_i \in I(P)$ for $i, j \in \{0, \ldots, n\}$, as they all vanish at $P$. For the reverse inclusion, let $i \in \{0, \ldots, n\}$ such that $a_i \neq 0$. For every $j \in \{0, \ldots, n\}$, let $b_j = \frac{a_j}{a_i}$. Let $U_i = \mathbb{P}^n(k) \smallsetminus V(x_i)$. Then $P = (b_0 : \cdots : b_n)$ and the dehomogenization of $P$ under the embedding $\mathbb{A}^n(k) \cong U_i \subseteq \mathbb{P}^n(k)$ is $P_* = (b_0, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n)$. Let's denote by $k[u_0, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n]$ the coordinate ring of $\mathbb{A}^n(k)$. By part (b) we know that $I(P_*)$ is the maximal ideal generated by $x_j - b_j$ for $j \in \{0, \ldots, i-1, i+1, \ldots, n\}$. Since $P$ doesn't intersect $V(x_i)$ we have $(P_*)^* = P$. Then $I(P) = I((P_*)^*) = I(P_*)^*$ is generated by $x_j - b_j x_i$ for $j \in \{0, \ldots, n\}$. Note that $a_i(x_j - b_j x_i) = a_i x_j - a_j x_i$. Thus

$$I(P) = (a_i x_0 - a_0 x_i, \ldots, a_i x_n - a_n x_i) \subseteq (\{a_i x_j - a_j x_i : i, j \in \{0, \ldots, n\}\}).$$

1 point for the first inclusion
1 point for $V(\{a_i x_j - a_j x_i : i, j \in \{0, \ldots, n\}\}) = \{P\}$
1 point for the dehomogenization
1 point for using part (b)
1 point for the homogenization
1 point for concluding the reverse inclusion.

(3) Let $X \subseteq \mathbb{A}^n(k)$ be a nonempty affine algebraic set such that $X \neq \mathbb{A}^n(k)$. Let $U_0 = \mathbb{P}^n \smallsetminus V(x_0)$, and $\varphi : \mathbb{A}^n(k) \to U_0 \subseteq \mathbb{P}^n(k)$ be the corresponding inclusion.
(a) (6 points) Define $X^*$. Show that $X^*$ is the smallest projective algebraic set containing $\varphi(X)$.
(b) (6 points) Show that $V(x_i) \not\subseteq X^*$ and that no irreducible component of $X^*$ is contained in $V(x_i)$.

**Solution**

(a) By definition $X^*$ is a projective algebraic set containing $\varphi(X)$. Let $Y$ be another projective algebraic set containing $\varphi(X)$. Then $X = \varphi^{-1}(\varphi(X)) \subseteq \varphi^{-1}(Y) = Y_*$. Thus $X^* \subseteq (Y_*)^* \subseteq Y$.
1 point for the definition
1 point for $\varphi(X) \subseteq X^*$

1 point for $X = \varphi^{-1}(\varphi(X))$
1 point for $\varphi^{-1}(Y) = Y_*$
1 point for deducing $X^* \subseteq (Y_*)^*$
1 point for $(Y_*)^* \subseteq Y$

(b) Without loss of generality we can assume that $X$ is irreducible. If $V(x_0) \subseteq X^*$, then $I(X)^* \subseteq I(X^*) \subseteq (x_0) \subseteq k[x_0, \dots, x_n]$. But if $f \in I(X)$ such that $f \neq 0$ (such an $f$ exists as $X \neq \mathbb{A}^n(k)$), then $f^* \notin (x_0)$ by construction of $f^*$. This is a contradiction, hence $V(x_0) \not\subseteq X^*$. Since $X^* \cap U_0 = \varphi(X) \neq \emptyset$ as $X \neq \emptyset$, then $X^* \not\subseteq \mathbb{P}^n(k) \setminus U_0 = V(x_0)$.
1 point for $I(X)^* \subseteq I(X^*)$
1 point for $I(X^*) \subseteq (x_i)$
1 point for using $X \neq \mathbb{A}^n(k)$
1 point for $f^* \notin (x_0)$
1 point for $X^* \cap U_0 = \varphi(X)$
1 point for using $X \neq \emptyset$

(4) Consider the projective plane curve $X = V(f)$ given by
$$f = x_0^5 + (x_1 + x_2)^2(x_0x_2^2 - x_2^3).$$

(a) (2 points) Compute a change of coordinates $\varphi : \mathbb{P}^2(k) \to \mathbb{P}^2(k)$ such that $\varphi(0 : 0 : 1) = (0 : 1 : -1)$, $\varphi(0 : 1 : 0) = (0 : 1 : 0)$ and $\varphi(V(x_0)) = V(x_0)$.

(b) (16) Compute the multiple points for $f$. Compute multiplicites, tangent lines and multiplicities of the tangent lines at each multiple point for $f$.

**Solution**

(a) The morphism $\varphi : \mathbb{P}^2(k) \to \mathbb{P}^2(k)$ given by $\varphi(x_0 : x_1 : x_2) = (x_0 : x_1 + x_2 : -x_2)$ is a change of coordinates because it is an isomorphism (with inverse $\varphi^{-1}(x_0 : x_1 : x_2) = (x_0 : x_1 + x_2 : -x_2)$), and it satisfies the required conditions as $x_0 \circ \varphi^{-1} = x_0$.
1 point for the correct linear forms and for checking the required conditions
1 point for the definition of change of coordinate

(b) The multiple points for $f$ can be computed as solutions of the linear system of equations $f = \frac{\partial f}{\partial x_0} = \frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = 0$. We compute the partial derivatives of $f$:
$$\frac{\partial f}{\partial x_0} = 5x_0^4 + (x_1 + x_2)^2 x_2^2,$$
$$\frac{\partial f}{\partial x_1} = 2(x_1 + x_2)(x_0x_2^2 - x_2^3),$$
$$\frac{\partial f}{\partial x_2} = 2(x_1 + x_2)(x_0x_2^2 - x_2^3) + (x_1 + x_2)^2(2x_0x_2 - 3x_2^2).$$
If $x_1 + x_2 = 0$, $f = 0$ gives $x_0 = 0$, and we get the multiple point $(0 : 1 : -1)$. If $x_1 + x_2 \neq 0$, $\frac{\partial f}{\partial x_1} = 0$ is equivalent to $(x_0x_2^2 - x_2^3) = 0$. Since $f = 0$, we get $x_0 = 0$. Then the equation $(x_0x_2^2 - x_2^3) = 0$ gives the multiple point $(0 : 1 : 0)$. If we dehomogenize $f$ with respect to $x_1$, we get $m_{(0:1:0)}(f) = m_{(0,0)}(x_0^5 + (1 + x_2)^2(x_0x_2^2 - x_2^3)) = 3$ and the tangent lines at $(0 : 1 : 0)$

are given by $x_0x_2^2 - x_2^3 = 0$. Thus at $(0:1:0)$ there are two tangent lines: $V(x_0 - x_2)$ of multiplicity 1, and $V(x_2)$ of multiplicity 2. For the multiplicity of $f$ at $(0:1:-1)$ we first compute

$$f \circ \varphi = x_0^5 + x_1^2(x_0x_2^2 + x_2^3).$$

If we dehomogenize $f \circ \varphi$ at $x_2$, we get

$$m_{(0:1:1)}(f) = m_{(0:0:1)}(f \circ \varphi) = m_{(0,0)}(x_0^5 + x_1^2(x_0 + 1)) = 2,$$

and $f \circ \varphi$ has only one tangent line at $(0:0:1)$: $V(x_1)$ of multiplicity 2. Thus $f$ has one tangent line of multiplicity 2 at $(0:1:-1)$: $V(x_1 + x_2)$.

3 points for computing the multiple points

6 points for $(0:1:0)$ distributed as follows: 1 for dehomogenization, 1 for multiplicity, 2 for tangent lines, 2 points for multiplicities of the tangent lines

1 point for $f \circ \varphi$

1 point for $m_{(0:1:-1)}(f) = m_{(0:0:1)}(f \circ \varphi)$

1 point for the dehomogenization of $f \circ \varphi$

1 point for the multiplicity

1 point for the tangent line to $f \circ \varphi$

1 point for the tangent line to $f$

1 point for multiplicity of tangent line.

(5) Let $f \in k[x, y]$ be a nonconstant polynomial and $l \in k[x, y]$ a polynomial of degree 1. Assume that $V(l)$ is tangent to $V(f)$ at a point $P \in V(f)$.

(a) (6 points) Show that $I(P, f \cap l) > m_p(f)$.

(b) (4 points) Assume that $I(P, f \cap l) = 3$, and that $V(l)$ is the unique tangent line to $V(f)$ at $P$. Show that $P$ is contained in a unique irreducible component of $V(f)$.

(c) (4 points) Show that there exists no projective plane curve of degree 7 that is tangent to the same line at five distinct points.

**Solution**

(a) Up to a change of coordinates, we can assume that $P = (0, 0)$. Write $f = \sum_{i=0}^{d} f_i$ with $f_i$ homogeneous of degree $i$ for each $i \in \{0, \dots, d\}$. Let $a = m_P(f)$. Then $a \geq 1$ as $P \in V(f)$ and $f_0 = \dots = f_{a-1} = 0$ and $f_a \neq 0$ as polynomials. Since $V(l)$ is tangent to $V(f)$ at $P$ and $l$ is an irreducible polynomial of degree 1, we have $l \mid f_a$. Thus

$$I(P, f \cap l) = I(P, (f - f_a) \cap l) \geq m_P(f - f_a)m_P(l) \geq a + 1,$$

as $m_p(f - f_a) = m_P(f_{a+1} + \dots + f_d) \geq a + 1$ and $m_P(l) = 1$.

2 points for correct use of the definition of $m_p(f)$

1 point for $I(P, f \cap l) = I(P, (f - f_a) \cap l)$

1 point for $I(P, (f - f_a) \cap l) \geq m_P(f - f_a)m_P(l)$

1 point for $m_p(f - f_a) \geq a + 1$

1 point for $m_P(l) = 1$

(b) If $V(f)$ has two irreducible components passing through $P$, then $f = gh$ for two polynomials $g, h \in k[x, y]$ such that $g(P) = h(P) = 0$. Hence, $m_P(g) \geq 1$, $m_P(g) \geq 1$, and $I(P, g \cap l) \geq 2$ and $I(P, h \cap l) \geq 2$ by part (a). Thus $I(P, f\cap l) = I(P, g\cap l) + I(P, h\cap l) \geq 4$ which contradicts the assumption $I(P, f\cap l) = 3$.

1 point for setting up $g$ and $h$

1 point for $m_P(g) \geq 1$, $m_P(g) \geq 1$

1 point for use of part (a)

1 point for $I(P, f\cap l) = I(P, g\cap l) + I(P, h\cap l)$

(c) Let $f, l \in k[x, y, z]$ be a nonconstant homogeneous polynomial, $\deg l = 1$. Let $P_1, \ldots, P_5$ distinct points in $V(f)\cap V(l) \subseteq \mathbb{P}^2(k)$. By Bezout's theorem we have $\deg f = \deg f \deg l \geq \sum_{i=1}^{5} I(P_i, f\cap l)$. If $V(f)$ is tangent to $V(l)$ at $P_1, \ldots, P_5$, then $I(P_i, f\cap l) \geq 2$ for all $i \in \{1, \ldots, 5\}$ and hence $\deg f \geq 10$. Thus there is no projective plane curve of degree 7 tangent to a line at five distinct points.

1 point for correct use of definition of projective plane curve/setting up notation

1 point for using Bezout's theorem

1 point for using part (a)

1 point for conclusion.

(6) Let $X$ and $Y$ be two affine algebraic sets.

(a) (9 points) State and prove the correspondence between polynomial maps $X \to Y$ and homomorphisms of coordinate rings.

(b) (2 points) State the correspondence between dominant rational maps $X \dashrightarrow Y$ and homomorphisms of function fields. Include all necessary definitions.

**Solution**

(a) Statement: Every polynomial map $\varphi : X \to Y$ induces a homomorphism of $k$-algebras $\widetilde{\varphi} : \Gamma(Y) \to \Gamma(X)$ by composition by $\varphi$. This construction induces a bijection between the set of polynomial maps $X \to Y$ and the set of homomorphisms of $k$-algebras $\Gamma(Y) \to \Gamma(X)$.

Proof: Let $X \subseteq \mathbb{A}^n(k)$ and $Y \subseteq \mathbb{A}^m(k)$ be alegbraic sets. A polynomial map $\varphi : X \to Y$ is given in coordinates by $\varphi = (\varphi_1, \ldots, \varphi_m)$ where $\varphi_1, \ldots, \varphi_m \in k[x_1, \ldots, x_n]$. Then composition by $\varphi$ is the $k$-algebra homomorphism

$$\Gamma(Y) = k[x_1, \ldots, x_m]/I(Y) \to \Gamma(X) = k[x_1, \ldots, x_n]/I(X), \qquad \overline{x}_i \mapsto \overline{\varphi}_i.$$

Let $\alpha : \Gamma(Y) \to \Gamma(X)$ be a homomorphism of $k$-algebras. For every $i \in \{1, \ldots, m\}$ let $\psi_i \in k[x_1, \ldots, x_n]$ such that $\alpha(\overline{x}_i) = \overline{\psi_i}$. Consider the polynomial map $\psi = (\psi_1, \ldots, \psi_m) : \mathbb{A}^n(k) \to \mathbb{A}^m(k)$. Since $\widetilde{\psi}$ factors by $\alpha$ by construction, we have $\widetilde{\psi}(I(Y)) \subseteq I(X)$ and hence $\psi(X) \subseteq Y$. Let $\varphi = \psi|_X : X \to Y$. Then $\varphi$ is a polynomial map such that $\widetilde{\varphi} = \alpha$ by construction. For the converse composition. Start with a polynomial map $\varphi = (\varphi_1, \ldots, \varphi_m) : X \to Y$, then $\widetilde{\varphi} : \Gamma(Y) \to \Gamma(X)$ is given by $\widetilde{\varphi}(\overline{x}_i) = \overline{\varphi}_i$. Choose possibly different $\psi_1, \ldots \psi_m \in k[x_1, \ldots, x_n]$ such that $\overline{\psi}_i = \widetilde{\varphi}(\overline{x}_i)$ for all $i \in \{1, \ldots, m\}$. Let $\psi = (\psi_1, \ldots, \psi_m) : X \to Y$. Then $\psi = \varphi$ as functions because $\varphi_i - \psi_i \in I(X)$ for all $i \in \{1, \ldots, m\}$.

2 points for statement

1 point for correct definition of polynomial map

1 point for proving composition by $\varphi$ is a homomorphism of $k$-algebras
1 point for choosing $\psi_i$
1 point for constructing $\psi$
1 point for constructing $\varphi$
1 point for one composition
1 point for the converse composition

(b) Let $X \subseteq \mathbb{A}^n(k)$ and $Y \subseteq \mathbb{A}^m(k)$ be algebraic sets. A rational map $\varphi : X \to Y$ is given in coordinates by $\varphi = (\varphi_1, \ldots, \varphi_m)$ for suitable rational functions $\varphi_1, \ldots, \varphi_m \in k(X)$. Then $\varphi$ induces a homomorphism of $k$-algebras $\widetilde{\varphi} : k(Y) \to k(X)$ given by $\widetilde{\varphi}(\overline{x}_i) = \varphi_i$.
2 points

(7) Consider the curve $C = V(y^2 - x^3 + x, y^3 - z^2) \subseteq \mathbb{A}^3(k)$. We denote by $\overline{x}, \overline{y}, \overline{z} \in k[x, y, z]/I(C)$ the classes of the variables $x, y, z$.
(a) (2 points) Show that the element $\frac{1}{\overline{y}} \in k(C)$ is contained in the subfield $k(\overline{x}, \overline{z})$.
(b) (7 points) Determine a plane curve $C' \subseteq \mathbb{A}^2(k)$ birational to $C$ and write down an explicit birational morphism $C' \dashrightarrow C$.

**Solution**

(a) In $k(C)$ we have
$$\frac{1}{\overline{y}} = \frac{\overline{y}^2}{\overline{y}^3} = \frac{\overline{x}^3 - \overline{x}}{\overline{z}^2}.$$
Hence, $\frac{1}{\overline{y}} \in k(\overline{x}, \overline{z})$.

(b) By part (a) we have $k(C) = k(\overline{x}, \overline{z})$. From the equations defining $C$ we have
$$\overline{z}^4 = \overline{y}^6 = (\overline{x}^3 - \overline{x})^3 = (\overline{x}^3 - \overline{x})^3.$$
The polynomial $z^4 - (x^3 - x)^3 \in k(x)[z]$ is irreducible as $(x^3 - x)^3$ is not a square in $k[x]$.
Let $C' = V(z^4 - (x^3 - x)^3) \subseteq \mathbb{A}^2(k)$. Then $k(C') = k(x)[z]/(z^4 - (x^3 - x)^3)$, and we have an isomorphism $k(x)[z]/(z^4 - (x^3 - x)^3) \to k(C)$ induced by the $k$-algebra homomorphism
$$k[x, z] \to k(C), \qquad x \mapsto \overline{x}, \ z \mapsto \overline{z}.$$
This isomorphism corresponds to the birational map
$$C \dashrightarrow C', \qquad (x, y, z) \mapsto (x, z),$$
which has inverse
$$C' \dashrightarrow C, \qquad (x, z) \mapsto \left(x, \frac{z^2}{x^3 - x}, z\right).$$

1 point for $k(C) = k(\overline{x}, \overline{z})$ from part (a)
1 point for finding a polynomial
1 point for irreducibility
1 point for birational iff isomorphic function fields
1 point for $k(C') \cong k(\overline{x}, \overline{z})$
1 point for $k(C) \cong k(\overline{x}, \overline{z})$
1 point for the explicit birational map.

The grade of the exam is computed as: $E =$ (number of points)/6.8

Hand-in component of the grade is computed as: $H =$ (number of points)/4

The final grade of the course is computed as: $\max\{E, 80\%E + 20\%H\}$. In order to pass the course the value $E \geq 5.5$ is required.