Exam WISB326, June 28, 2022, 17:00-20:00

Exam problems

- Let k be an algebraically closed field of characteristic 0.

(1) (12 points) Let $X = V((x_1+x_2)(2x_1+x_3), (x_2-x_3^2)(4x_1^2-x_3^2)) \subseteq \mathbb{A}^3(k)$. Determine the irreducible components of X.

(2) Let $X \subseteq \mathbb{A}^n(k)$ be an affine algebraic set.

(a) (4 points) Show that $X = V(I(X))$.

(b) (4 points) Show that X is a point if and only if $I(X)$ is a maximal ideal.

(c) (6 points) Let $P = (a_0 : \cdots : a_n) \in \mathbb{P}^n(k)$. Show that the ideal $I(P) \subseteq k[x_0, \dots, x_n]$ is generated by the polynomials $a_i x_j - a_j x_i$ for $i, j \in \{0, \dots, n\}$.

(3) Let $X \subseteq \mathbb{A}^n(k)$ be a nonempty affine algebraic set such that $X \neq \mathbb{A}^n(k)$. Let $U_0 = \mathbb{P}^n \smallsetminus V(x_0)$, and $\varphi : \mathbb{A}^n(k) \to U_0 \subseteq \mathbb{P}^n(k)$ be the corresponding inclusion.

(a) (6 points) Define X^*. Show that X^* is the smallest projective algebraic set containing $\varphi(X)$.

(b) (6 points) Show that $V(x_i) \nsubseteq X^*$ and that no irreducible component of X^* is contained in $V(x_i)$.

(4) Consider the projective plane curve $X = V(f)$ given by

$$f = x_0^5 + (x_1 + x_2)^2(x_0 x_2^2 - x_2^3).$$

(a) (2 points) Compute a change of coordinates $\varphi : \mathbb{P}^2(k) \to \mathbb{P}^2(k)$ such that $\varphi(0 : 0 : 1) = (0 : 1 : -1)$, $\varphi(0 : 1 : 0) = (0 : 1 : 0)$ and $\varphi(V(x_0)) = V(x_0)$.

(b) (16 points) Compute the multiple points for f. Compute multiplicites, tangent lines and multiplicities of the tangent lines at each multiple point for f.

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(5) Let $f \in k[x, y]$ be a nonconstant polynomial and $l \in k[x, y]$ a polynomial of degree 1. Assume that $V(l)$ is tangent to $V(f)$ at a point $P \in V(f)$.

(a) (6 points) Show that $I(P, f \cap l) > m_p(f)$.

(b) (4 points) Assume that $I(P, f \cap l) = 3$, and that $V(l)$ is the unique tangent line to $V(f)$ at P. Show that P is contained in a unique irreducible component of $V(f)$.

(c) (4 points) Show that there exists no projective plane curve of degree 7 that is tangent to the same line at five distinct points.

(6) Let X and Y be two affine algebraic sets.

(a) (9 points) State and prove the correspondence between polynomial maps $X \to Y$ and homomorphisms of coordinate rings.

(b) (2 points) State the correspondence between dominant rational maps $X \dashrightarrow Y$ and homomorphisms of function fields. Include all necessary definitions.

(7) Consider the curve $C = V(y^2 - x^3 + x, y^3 - z^2) \subseteq \mathbb{A}^3(k)$. We denote by $\overline{x}, \overline{y}, \overline{z} \in k[x, y, z]/I(C)$ the classes of the variables x, y, z.

(a) (2 points) Show that the element $\frac{1}{\overline{y}} \in k(C)$ is contained in the subfield $k(\overline{x}, \overline{z})$.

(b) (7 points) Determine a plane curve $C' \subseteq \mathbb{A}^2(k)$ birational to C and write down an explicit birational morphism $C' \dashrightarrow C$.

Success!