Retake exam WISB326, July 12, 2022, 17:00-20:00
Solutions and grading scheme

Exam problems

- Let k be an algebraically closed field of characteristic 0.

(1) (12 points) Let $X = V(x_1 + x_2 - x_3^3) \subseteq \mathbb{A}^3(k)$. For each of the following sets determine whether it is an open subset of X and whether it is a closed subset of X:

$$(X \cap V(x_1)) \smallsetminus \{(0,0,1)\}, \qquad (X \cap V(x_1^2 - x_2^2 - x_3^3(x_1 - x_2))) \smallsetminus \{(0,1,1)\},$$
$$(X \cap V(x_1 - x_2, x_2 - 2x_3)) \smallsetminus \{(0,0,0)\}.$$

Solution

Since $(0,0,1) \notin X$, we see that $(X \cap V(x_1)) \smallsetminus \{(0,0,1)\} = X \cap V(x_1)$ is an intersection of two algebraic sets, hence it is an algebraic set in $\mathbb{A}^3(k)$, and hence, a closed subset of X. Since $k[x_1, x_2, x_3]/(x_1 + x_2 - x_3^3) \cong k[x_2, x_3]$ is an integral domain, X is irreducible, and hence every nonempty open subset is dense in X. We see that $(0,0,0) \in X \cap V(x_1)$, thus $X \cap V(x_1)$ is nonempty. Since $X \cap V(x_1)$ is closed in X and $(0,0,1) \notin X \cap V(x_1)$, the complement of $X \cap V(x_1)$ is a nonempty open subset of X, and it doesn't intersect $X \cap V(x_1)$. Thus $X \cap V(x_1)$ is not dense in X, and hence it is not an open subset of X.

We observe that $X \subseteq V(x_1^2 - x_2^2 - x_3^3(x_1 - x_2))$, as $V(x_1^2 - x_2^2 - x_3^3(x_1 - x_2)) = V(x_1 - x_2) \cup V(x_1 + x_2 - x_3^3)$. Thus $(X \cap V(x_1^2 - x_2^2 - x_3^3(x_1 - x_2))) \smallsetminus \{(0,1,1)\} = X \smallsetminus \{(0,1,1)\}$ is an open subset of X because its complement $\{(0,1,1)\}$, which is a point, is an algebraic set. Since $(0,0,0) \in X \smallsetminus \{(0,1,1)\}$, we have that $X \smallsetminus \{(0,1,1)\}$ is a nonempty open subset of X, and hence dense in X. Thus X is the smallest algebraic set that contains $X \smallsetminus \{(0,1,1)\}$, hence $X \smallsetminus \{(0,1,1)\}$ is not an algebraic set, and in particular, it is not a closed subset of X.

We observe that $X \cap V(x_1 - x_2, x_2 - 2x_3) = \{(0,0,0), (4,4,2), (-4,-4,-2)\}$, and hence $(X \cap V(x_1 - x_2, x_2 - 2x_3)) \smallsetminus \{(0,0,0)\} = \{(4,4,2), (-4,-4,-2)\}$ is a finite set of points, thus an algebraic set and hence a closed subset of X. If $\{(4,4,2), (-4,-4,-2)\}$ was an open subset of X, it would be dense, because X is irreducible. But this is not possible because the complement of $\{(4,4,2), (-4,-4,-2)\}$ in X is a nonempty open subset of X that does not intersect $\{(4,4,2), (-4,-4,-2)\}$. Thus $(X \cap V(x_1 - x_2, x_2 - 2x_3)) \smallsetminus \{(0,0,0)\}$ is not open in X.

4 points for $(X \cap V(x_1)) \smallsetminus \{(0,0,1)\}$
4 points for $(X \cap V(x_1^2 - x_2^2 - x_3^3(x_1 - x_2))) \smallsetminus \{(0,1,1)\}$
4 points for $(X \cap V(x_1 - x_2, x_2 - 2x_3)) \smallsetminus \{(0,0,0)\}$

(2) Let $X, Y \subseteq \mathbb{A}^n(k)$ be two algebraic sets.
  (a) (3 points) Define $I(X)$. Show that $I(X \cup Y) = I(X) \cap I(Y)$.
  (b) (5 points) Assume that $n = 2$, $X = \{(0,t) : t \in k\}$ and $Y = \{(1,0)\}$. Show that $I(X \cup Y) = I(X)I(Y)$.

Solution

(a) $I(X)$ is the set of polynomials vanishing at all points in X. A polynomial f belongs to $I(X \cup Y)$ if and only if it vanishes at all points in $X \cup Y$, which holds if and only if f vanishes at all points in X and at all points in Y, which is equivalent to $f \in I(X)$ and $f \in I(Y)$.
1 point for definition
1 point for one inclusion
1 point for reverse inclusion

(b) Let $k[x, y]$ be the coordinate ring of $\mathbb{A}^2(k)$. Then $I(X) = (x)$ and $I(Y) = (x - 1, y)$. By part (a) we know that $I(X \cup Y) = I(X) \cap I(Y)$. We know that $I(X)I(Y) \subseteq I(X) \cap I(Y)$, as if $f \in I(X)$ and $g \in I(Y)$, then $fg \in I(X) \cap I(Y)$. For the reverse inclusion, let $f \in I(X) \cap I(Y)$. Since $f \in I(X)$ we have $x \mid f$. Thus $f = xg$ for some $g \in k[x, y]$. Observe that $x \notin I(Y)$ as $Y \not\subseteq X$. Since $f \in I(Y)$, $I(Y)$ is a prime ideal and $x \notin I(Y)$, we get $g \in I(Y)$. Thus $f \in I(X)I(Y)$.
1 point for computation of ideals $I(X)$ and/or $I(Y)$
1 point for $I(X)I(Y) \subseteq I(X) \cap I(Y)$
1 point for proving $x \notin I(Y)$
1 point for using $I(X)$ principal
1 point for using $I(Y)$ prime

(3) (12 points) State and prove the projective Nullstellensatz, both the case of the empty set and the case of the nonempty projective algebraic sets (you can use the affine versions without proof).

Solution Statement: Let $I \subseteq k[x_0, \dots, x_n]$ be a homogenous ideal, and $X = V(I) \subseteq \mathbb{P}^n(k)$. Then

(a) $X = \emptyset$ if and only if $(x_0, \dots, x_n)^N \subseteq I$ for some $N \geq 1$.

(b) If $X \neq \emptyset$, then $I(X) = \sqrt{I}$.

Proof: Let $Y = V(I) \subseteq \mathbb{A}^{n+1}(k)$. We have $\sqrt{I} = I(Y)$ by the affine Nullstellensatz.

(a) $(x_0, \dots, x_n)^N \subseteq I$ holds for some $N \geq 1$ if and only if $(x_0, \dots, x_n) \subseteq \sqrt{I} = I(Y)$, which is equivalent to $Y \subseteq \{(0, \dots, 0)\}$ in $\mathbb{A}^{n+1}(k)$. The latter is equivalent to $X = \emptyset$.

(b) If $X \neq \emptyset$, then Y is the affine cone over X. We have $I(X) = I(Y) = \sqrt{I}$, where the first equality holds because X is nonempty.

2 points for correct statement (a)
2 points for correct statement (b)
1 point for passing to the radical
1 point for using affine Nullstellensatz
1 point for $Y \subseteq \{(0, \dots, 0)\}$
1 point for conclusion
1 point for Y affine cone over X
1 point for $I(X) = I(Y)$
1 point for use of affine Nullstellensatz
1 point for conclusion or for using $X \neq \emptyset$.

(4) Let $GL_{n+1}(k)$ be the set of invertible $(n+1)\times(n+1)$-matrices with entries in k. For every matrix $A \in GL_{n+1}(k)$, let $\varphi_A : \mathbb{P}^n(k) \dashrightarrow \mathbb{P}^n(k)$ be the rational map that sends a point with homoegenous coordinates $(x_0 : \cdots : x_n)$ to the point represented by the coordinates $A\begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix}$.

(a) (2 points) Show that φ_A is a projective change of coordinates.

(b) (10 points) Show that there is a bijection between the set of invertible matrices $A \in GL_{n+1}(k)$ with $\det A = 1$ and the set of projective changes of coordinates on $\mathbb{P}^n(k)$.

(c) (6 points) Show that composition by $A\begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix}$ induces an isomorphism

$$\widetilde{\varphi}_A : k[x_0, \dots, x_n] \to k[x_0, \dots, x_n]$$

that sends homogeneous polynomials of degree d to homogeneous polynomials of degree d for all $d \geq 0$.

(d) (7 points) Let $f \in k[x, y, z]$ be a homogeneous polynomial of degree $d > 0$. Let $P \in V(g) \subseteq \mathbb{P}^2(k)$ be a point of multiplicity $m_P(f) = d$ for f. Show that $V(f) \subseteq \mathbb{P}^2(k)$ is a union of lines through P.

Solution.

(a) Since A is invertible, it has trivial kernel, and hence, φ_A is a morphism. Write $A = (a_{i,j})_{0\leq i,j\leq n}$ with $a_{i,j} \in k$ for all $i, j \in \{0, \dots, n\}$. For $i \in \{0, \dots, n\}$, let $\varphi_{A,i} = \sum_{j=0}^{n} a_{i,j}x_j$. Then $\varphi_A = (\varphi_{A,0} : \dots, \varphi_{A,n})$ is given by homogeneous polynomials of degree 1. Finally $\varphi_{A,0} : \dots, \varphi_{A,n}$ are lineary independent because A is invertible. Thus φ_A is a change of coordinates.

1 point for morphism

1 point for correct definition of change of coordinates

(b) By part (a) there is a function from $GL_{n+1}(k)$ to the set of changes of coordinates on $\mathbb{P}^n(k)$ given by $A \mapsto \varphi_A$. We first prove injectivity. Two invertible matrices A, B define the same change of coordinates if and only if there is a nonzero constant $\lambda \in k$ such that $A\begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix} = \lambda B\begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix}$ for all $(x_0, \dots, x_n) \in k^{n+1}$, i.e., if and only if $A = \lambda B$. Thus $\det A = \lambda^n \det B$. We get that A and B have both determinant 1 if and only if $A = B$. For surjectivity, let $\varphi : \mathbb{P}^n(k) \to \mathbb{P}^n(k)$ be a change of coordinates. Then there are linearly independent linear polynomials $\varphi_0, \dots, \varphi_n \in k[x_0, \dots, x_n]$ such that $\varphi = (\varphi_0 : \cdots : \varphi_n)$. Let A be the matrix corresponding to the linear map $k^{n+1} \to k^{n+1}$ that sends a vector $(v_0, \dots, v_n) \in k^{n+1}$ to $(\varphi_0(v_0, \dots, v_n), \dots, \varphi_n(v_0, \dots, v_n))$. Then A is invertible as $\varphi_0, \dots \varphi_n$ are linearly independent. Hence, $\det A \neq 0$. Let $a \in k$ such that $a^{n+1} = \det A$ (notice that a exists as k is algebraically closed). Then $\frac{1}{a}A$ is an invertible matrix of determinant 1 and $\varphi_{\frac{1}{a}A} = \varphi_A = \varphi$.

1 point for setting up the function

1 point for correct use of definition of change of coordinates
1 point for correct use of definition of homogeneous coordinates
1 point for $A = \lambda B$
1 point for $\det A = \lambda^n \det B$
1 point for concluding injectivity
1 point for defining A from φ
1 point for A invertible
1 point for existence of a
1 point for $\varphi_{\frac{1}{a}A} = \varphi_A = \varphi$

(c) For $i \in \{0, \dots, n\}$, let $\varphi_i := \widetilde{\varphi}_A(x_i)$, then φ_i is a linear polynomial by definition of $\widetilde{\varphi}_A$. Then $\widetilde{\varphi}_A$ is the k-algebra homomorphism defined by $x_i \mapsto \varphi_i$ for $i \in \{0, \dots, n\}$. Since A is invertible, $\widetilde{\varphi}_A \circ \widetilde{\varphi}_{A^{-1}}$ is the identity on $k[x_0, \dots, x_n]$, Hence φ_A is an isomorphism. Let $h = \prod_{i=0}^n x_i^{e_i}$ be a monomial of degree $d \geq 0$, then $\widetilde{\varphi}_A(h) = \prod_{i=0}^n \varphi_i^{e_i}$. Since $\varphi_0, \dots, \varphi_n$ are linear, $\widetilde{\varphi}_A(h)$ is a product of d linear polynomials, and hence a homogeneous polynomial of degree d. Notice that since $\widetilde{\varphi}_A$ is injective, $\widetilde{\varphi}_A(h) = 0$ if and only if $h = 0$. Every homogeneous polynomial f of degree d is a sum of monomials of degree d, hence $\widetilde{\varphi}_A(f)$ is a sum of homogeneous polynomials of degree d, and hence a homogeneous polynomial of degree d.
1 point for φ_i linear
1 point for $\widetilde{\varphi}_A$ ring homomorphism
1 point for $\widetilde{\varphi}_A$ invertible
2 points for preserving degree of monomials
1 point for preserving degree of polynomials

(d) By part (c) up to a change of coordinates we can assume without loss of generality that $P = (0 : 0 : 1)$. Let f_* be the dehomogenization of f with respect to z. Then f_* is a polynomial of degree at most d in $k[x, y]$ and $m_{(0,0)}(f_*) = m_Q(f) = d$. Thus f_* is a homogeneous polynomial of degree d, in particular, $f_* = f$. Without loss of generality we can assume that x appears in f. Let g be the dehomogenization of f_* with respect to y. Let $e = \deg g$. Then $e \leq d$, and $g \in k[x]$ is a product of e linear factors $g_1, \dots, g_e$ because k is algebraically closed. For $i \in \{1, \dots, e\}$ let f_i be the homogenization of g_i with respect to y. Then $f_1, \dots, f_e$ are linear polynomials and $f = y^{d-e} f_1 \cdots f_e$. Thus $V(f)$ is a union of lines through P: $V(f_1), \dots, V(f_e)$ and $V(y)$ if $d > e$.
1 point for reduction to $(0 : 0 : 1)$
2 points for f_* homogeneous of degree d
1 point for second dehomogenization
1 point for g product of linear factors
1 point for f_i linear
1 point for $f = y^{d-e} f_1 \cdots f_e$

(5) Consider the projective plane curves given by the polynomials

$$f = x_0^6 + x_0 x_1^5 + x_1^2 x_2^2 (2x_1 - 3x_2)^2.$$

(a) (6 points) Compute all the points in the intersection $V(f) \cap V(x_0)$ and for each point $P \in V(f) \cap V(x_0)$ compute the multiplicity $m_P(f)$.

(b) (5 points) Compute the intersection number $I((0:1:0), f \cap x_0)$.
(c) (4 points) For each point $P \in V(f) \cap V(x_0)$ compute the intersection number $I(P, f \cap x_0)$.

Solution.

(a) We solve the system of equations $f = x_0 = 0$, which is equivalent to the system of equations $x_1^2 x_2^2 (2x_1 - 3x_2)^2 = x_0 = 0$. Hence,

$$V(f) \cap V(x_0) = \{(0:0:1), (0:1:0), (0:3:2)\}.$$

A point $P \in V(f)$ satisfies $m_P(f) = 1$ if and only if P is a nonsingular point. We observe that the partial derivative $\frac{\partial f}{\partial x_0} = 6x_0^5 + x_1^5$ does not vanish at $(0:1:0)$ nor at $(0:3:2)$. Hence, $m_{(0:1:0)}(f) = m_{(0:3:2)}(f) = 1$. We dehomogenize f with respect to x_2 to get $f_* = x_0^6 + x_0 x_1^5 + x_1^2(2x_1 - 3)^2$ with lowest order term of degree 2. Hence, $m_{(0:0:1)}(f) = m_{(0,0)} f_* = 2$.

2 points for computing the intersection points
2 points for the multiplicities at $(0:1:0)$ and $(0:3:2)$
2 points for the multiplicity at $(0:0:1)$.

(b) By dehomogenizing with respect to x_1 we get

$$I((0:1:0), f \cap x_0) = I((0,0), f' \cap x_0)$$

where $f' = x_0^6 + x_0 + x_2^2(2 - 3x_2)^2$. Then

$$\begin{aligned} I((0,0), f_* \cap x_0) &= I((0,0), (x_2^2(2-3x_2)^2) \cap x_0) \\ &= m_{(0,0)}(x_2^2(2-3x_2)^2) m_{(0,0)}(x_0) = 2, \end{aligned}$$

as $x_2^2(2-3x_2)^2$ and x_0 have no common tangent lines at $(0,0)$ and $V(x_0)$ is nonsingular.

1 point for $I((0:1:0), f \cap x_0) = I((0,0), f' \cap x_0)$
1 point for dehomogenization
3 points to conclude

(c) The tangent lines at $(0:0:1)$ are given by $x_1^2 = 0$. Thus $V(f)$ and $V(x_0)$ have no common tangent lines at $(0:0:1)$. Thus $I((0:0:1), f \cap x_0) = m_{(0:0:1)}(f) m_{(0:0:1)}(x_0) = 2$, as $V(x_0)$ is smooth. By Bezout's theorem we know that $\sum_{P \in V(f) \cap V(x_0)} I(P, f \cap x_0) = \deg f \deg x_0 = 6$. Hence,

$$I((0:3:2), f \cap x_0) = 6 - I((0:1:0), f \cap x_0) - I((0:0:1), f \cap x_0) = 2.$$

2 points for $I((0:1:0), f \cap x_0)$
2 points for the correct application of Bezout's theorem

(6) Consider the morphism

$$\varphi : \mathbb{A}^1(k) \to \mathbb{P}^3(k), \qquad t \mapsto (1 : t^2 : t^3 : t^5),$$

and let C be the Zariski closure of $\varphi(\mathbb{A}^1(k))$.

(a) (4 points) Show that C is irreducible.
(b) (4 points) Show that C is a curve.
(c) (7 points) Show that the morphism $\varphi : \mathbb{A}^1 \to C$ is birational and write down an explicit inverse map.
(d) (3 points) Show that $\varphi(\mathbb{A}^1(k)) \subseteq \mathbb{P}^3(k)$ is not a projective algebraic set.

Solution.

(a) We observe that $\varphi(\mathbb{A}^1(k))$ is dense in C by definition of C. Assume that C is not irreducible, and write $C = C_1 \cap C_2$ with C_1, C_2 proper closed subsets of C. Then $\varphi(\mathbb{A}^1(k)) \cap C_i \supseteq \varphi(\mathbb{A}^1(k)) \cap (C_i \smallsetminus C_j) \neq \emptyset$ for $i, j \in \{1, 2\}$, $i \neq j$. Then $\varphi^{-1}(C_1) \cup \varphi^{-1}(C_2) = \mathbb{A}^1(k)$, and $\varphi^{-1}(C_1)$ and $\varphi^{-1}(C_2)$ are both proper closed subsets of $\mathbb{A}^1(k)$. This contradicts the fact that $\mathbb{A}^1(k)$ is irreducible. Thus we conclude that C is irreducible.
1 point for density
1 point for definition of irreducibility
1 point for $\varphi(\mathbb{A}^1(k)) \cap C_i \neq \emptyset$
1 point for using continuity

(b) Since C is irreducible by (a), it suffices to show that C has dimension 1. The rational map $\varphi : \mathbb{A}^1 \to C$ is dominant, hence it induces an inclusion of function fields $k(C) \subseteq k(\mathbb{A}^1(k))$. Since $k(\mathbb{A}^1(k)) = 1$ has transcendent degree 1 over k, we deduce that $k(C)$ has transcendent degree 0 or 1 over k. If $k(C)$ had transcendent degree 0 over k, then C would be a point, and in particular φ would be a constant map, which is not the case. Thus $k(C)$ had transcendent degree 1, and C has dimension 1.
1 point for definition of curve
1 point for inclusion of function fields
1 point for $\mathrm{trdeg}_k k(C) \in \{0, 1\}$
1 point for excluding $\mathrm{trdeg}_k k(C) = 0$

(c) Consider the rational map

$$\psi : C \dashrightarrow \mathbb{A}^1(k), \qquad (x_0 : x_1 : x_2 : x_3) \mapsto \frac{x_2}{x_1}.$$

Then $\psi \circ \varphi$ represents the identity on $\mathbb{A}^1(k)$ as a rational map, and ψ induces the inclusion of function fields

$$\widetilde{\psi} : k(\mathbb{A}^1(k)) = k(t) \subseteq k(C), \qquad t \mapsto \frac{\overline{x}_2}{\overline{x}_1}.$$

It suffices to show that $\widetilde{\psi}$ is surjective. We recall that $k(C)$ is the set of quotients of classes of homogeneous polynomials in $k[x_0, \ldots, x_3]/I(C)$ with numerator and denominator of the same degree. In particular, $k(C)$ is generated by $\frac{\overline{x}_1}{\overline{x}_0}, \frac{\overline{x}_2}{\overline{x}_0}, \frac{\overline{x}_3}{\overline{x}_0}$ over k. We observe that $x_0x_3 - x_1x_2$ vanishes on $\varphi(\mathbb{A}^1(k))$, and hence on C. Hence $\frac{\overline{x}_3}{\overline{x}_0} = \frac{\overline{x}_1}{\overline{x}_0}\frac{\overline{x}_2}{\overline{x}_0}$ in $k(C)$. Also the polynomial $x_0x_2^2 - x_1^3$ vanishes on $\varphi(\mathbb{A}^1(k))$, and hence on C. Thus

$$\frac{\overline{x}_1}{\overline{x}_0} = \frac{\overline{x}_1^3}{\overline{x}_0\overline{x}_1^2} = \left(\frac{\overline{x}_2}{\overline{x}_1}\right)^2, \qquad \frac{\overline{x}_2}{\overline{x}_0} = \frac{\overline{x}_2^3}{\overline{x}_0\overline{x}_2^2} = \left(\frac{\overline{x}_2}{\overline{x}_1}\right)^3.$$

Thus $k(C) = k\left(\frac{\overline{x}_2}{\overline{x}_1}\right)$ is the image of $\widetilde{\psi}$.
1 point for the explicit inverse
1 point for explicit inclusion of function fields
1 point for birational iff same function field, or for definition of birational map
1 point for definition of function field of a projective variety
3 points for surjectivity of $\widetilde{\psi}$.

(d) If $\varphi(\mathbb{A}^1(k))$ were a projective algebraic set, then $\varphi(\mathbb{A}^1(k))$ would be its own closure. Thus it suffices to show that $\varphi(\mathbb{A}^1(k)) \neq C$. We observe that C is the closure of the set

$$\left\{\left(1 : \frac{x_2^2}{x_1^2} : \frac{x_2^3}{x_1^3} : \frac{x_2^5}{x_1^5}\right) : x_1, x_2 \in k^\times\right\} = \left\{\left(x_1^5 : x_1^3x_2^2 : x_1^2x_2^3 : x_2^5\right) : x_1, x_2 \in k^\times\right\},$$

and hence C contains the point $(0 : 0 : 0 : 1)$ (obtained by taking $x_1 = 0, x_2 = 1$ in the second description). Since $(0 : 0 : 0 : 1)$ is not in $\varphi(\mathbb{A}^1(k))$, we conclude that $\varphi(\mathbb{A}^1(k))$ is not a projective algebraic set.

1 point for reduction to showing $\varphi(\mathbb{A}^1(k)) \neq C$.

2 points for $(0 : 0 : 0 : 1) \in C$.

The grade of the retake exam is computed as: $E = (\text{number of points})/6.8$

Hand-in component of the grade is computed as: $H = (\text{number of points})/4$

The final grade of the course is computed as: $\max\{E, 80\%E + 20\%H\}$. In order to pass the course the value $E \geq 5.5$ is required.