Retake exam WISB326, July 12, 2022, 17:00-20:00

Exam problems

- Let k be an algebraically closed field of characteristic 0.

(1) (12 points) Let $X = V(x_1 + x_2 - x_3^3) \subseteq \mathbb{A}^3(k)$. For each of the following sets determine whether it is an open subset of X and whether it is a closed subset of X:

$$(X \cap V(x_1)) \setminus \{(0,0,1)\}, \qquad (X \cap V(x_1^2 - x_2^2 - x_3^3(x_1 - x_2))) \setminus \{(0,1,1)\},$$
$$(X \cap V(x_1 - x_2, x_2 - 2x_3)) \setminus \{(0,0,0)\}.$$

(2) Let $X, Y \subseteq \mathbb{A}^n(k)$ be two algebraic sets.

(a) (3 points) Define $I(X)$. Show that $I(X \cup Y) = I(X) \cap I(Y)$.

(b) (5 points) Assume that $n = 2$, $X = \{(0,t) : t \in k\}$ and $Y = \{(1,0)\}$. Show that $I(X \cup Y) = I(X)I(Y)$.

(3) (12 points) State and prove the projective Nullstellensatz, both the case of the empty set and the case of the nonempty projective algebraic sets (you can use the affine versions without proof).

(4) Let $GL_{n+1}(k)$ be the set of invertible $(n+1) \times (n+1)$-matrices with entries in k. For every matrix $A \in GL_{n+1}(k)$, let $\varphi_A : \mathbb{P}^n(k) \dashrightarrow \mathbb{P}^n(k)$ be the rational map that sends a point with homoegenous coordinates $(x_0 : \cdots : x_n)$ to the point represented by the coordinates $A \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix}$.

(a) (2 points) Show that φ_A is a projective change of coordinates.

(b) (10 points) Show that there is a bijection between the set of invertible matrices $A \in GL_{n+1}(k)$ with $\det A = 1$ and the set of projective changes of coordinates on $\mathbb{P}^n(k)$.

(c) (6 points) Show that composition by $A \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix}$ induces an isomorphism

$$\widetilde{\varphi}_A : k[x_0, \ldots, x_n] \to k[x_0, \ldots, x_n]$$

that sends homogeneous polynomials of degree d to homogeneous polynomials of degree d for all $d \geq 0$.

Continues on the next page.

(d) (7 points) Let $f \in k[x, y, z]$ be a homogeneous polynomial of degree $d > 0$. Let $P \in V(f) \subseteq \mathbb{P}^2(k)$ be a point of multiplicity $m_P(f) = d$ for f. Show that $V(f) \subseteq \mathbb{P}^2(k)$ is a union of lines through P.

(5) Consider the projective plane curves given by the polynomials

$$f = x_0^6 + x_0x_1^5 + x_1^2x_2^2(2x_1 - 3x_2)^2.$$

(a) (6 points) Compute all the points in the intersection $V(f) \cap V(x_0)$ and for each point $P \in V(f) \cap V(x_0)$ compute the multiplicity $m_P(f)$.

(b) (5 points) Compute the intersection number $I((0:1:0), f \cap x_0)$.

(c) (4 points) For each point $P \in V(f) \cap V(x_0)$ compute the intersection number $I(P, f \cap x_0)$.

(6) Consider the morphism

$$\varphi : \mathbb{A}^1(k) \to \mathbb{P}^3(k), \qquad t \mapsto (1 : t^2 : t^3 : t^5),$$

and let C be the Zariski closure of $\varphi(\mathbb{A}^1(k))$.

(a) (4 points) Show that C is irreducible.

(b) (4 points) Show that C is a curve.

(c) (7 points) Show that the morphism $\varphi : \mathbb{A}^1 \to C$ is birational and write down an explicit inverse map.

(d) (3 points) Show that $\varphi(\mathbb{A}^1(k)) \subseteq \mathbb{P}^3(k)$ is not a projective algebraic set.

Success!