Measure and Integration: Solution Final Exam 2021-22

(1) Consider the measure space $([0,1], \mathcal{B}([0,1]), \lambda)$, where $\mathcal{B}([0,1])$ and λ are the restrictions of the Borel σ-algebra and Lebesgue measure to $[0,1]$. Let $u \in \mathcal{M}(\mathcal{B}([0,1]))$ be a bounded function that is continuous λ almost everywhere. Define $u_n(x) = u\Big(\frac{nx}{n+1}\Big)$ for $n \geq 1$. Prove that for any $p \in [1,\infty)$ one has

$$\lim_{n\to\infty} ||u_n||_p = ||u||_p.$$

(1.5 pt)

Proof : By continuity of u, we have $\lim_{n\to\infty} u_n(x) = u(x)$ λ a.e. Since u is bounded, there exists an $M \geq 0$ such that $|u| \leq M$ and hence $|u_n| \leq M$ for all n. Since $\lambda([0,1]) = 1$, we see that the constant function $M \in \mathcal{L}^p(\lambda)$ for all $p \geq 1$ implying that $u, u_n \in \mathcal{L}^p(\lambda)$ for all n. By Theorem 13.9, we have

$$\lim_{n\to\infty} ||u_n - u||_p = 0.$$

From the (reverse) triangle inequality, we have $|\,||u_n||_p - ||u||_p\,| \leq ||u_n - u||_p$, thus

$$\lim_{n\to\infty} ||u_n||_p = ||u||_p.$$

(2) Consider the measure space $((0,\infty), \mathcal{B}((0,\infty)), \lambda)$, where $\mathcal{B}((0,\infty))$ is the Borel σ-algebra restricted to $(0,\infty)$ and λ is the restriction of Lebesgue measure to $(0,\infty)$. Let $u(x) = \dfrac{x}{e^x - 1}$ for $x \in (0,\infty)$.

(a) Prove that $u \in \mathcal{L}^1((0,\infty), \lambda)$. (Hint: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$) (1.5 pts)

(b) Prove that $\lim_{n\to\infty} \int_{(0,\infty)} \frac{n}{e^x - 1} \sin(\frac{x}{n})\, d\lambda(x) = \int_{(0,\infty)} u(x)\, d\lambda(x)$.(Hint: $|\frac{\sin y}{y}| \leq 1$ for all y and $\lim_{y\to 0} \frac{\sin y}{y} = 1$.) (1.5 pts)

Proof (a): From $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, we see that $e^x - 1 \geq \frac{x^n}{n!}$ for all $n \geq 1$. Thus, $u(x)\mathbb{I}_{(0,1)}(x) \leq 1$ and $u(x)\mathbb{I}_{[1,\infty)} \leq \frac{6}{x^2}$. This shows that $u(x) = |u(x)| \leq \mathbb{I}_{(0,1)}(x) + \frac{6}{x^2}\mathbb{I}_{[1,\infty)}(x)$. Since the improper Riemann-integral $(R)\int_1^\infty \frac{1}{x^2}$ exists and is finite, we have by Corollary 12.11

$$\int_{(0,\infty)} u\, d\lambda \leq 1 + (R) \int_1^\infty \frac{6}{x^2}\, dx = 7 < \infty.$$

Hence, $u \in \mathcal{L}^1((0,\infty), \lambda)$.

Proof (b): Let $u_n(x) = \dfrac{n}{e^x - 1}\sin(\dfrac{x}{n})$. Using $|\dfrac{\sin y}{y}| \leq 1$ for all y and $\lim_{y\to 0}\dfrac{\sin y}{y} = 1$, we have $|u_n(x)| \leq \dfrac{x}{e^x - 1} = u(x)$ for all n and $\lim_{n\to\infty} u_n(x) = u(x)$. Since $u \in \mathcal{L}^1((0,\infty), \lambda)$, we have $u_n \in \mathcal{L}^1((0,\infty), \lambda)$ for all $n \geq 1$. So by the Lebesgue dominated convergence theorem,

$$\lim_{n\to\infty} \int_{(0,\infty)} \frac{n}{e^x - 1} \sin(\frac{x}{n})\, d\lambda(x) = \lim_{n\to\infty} \int_{(0,\infty)} u_n(x)\, d\lambda(x) = \int_{(0,\infty)} u(x)\, d\lambda(x).$$

(3) Let $(X, \mathcal{A}, \mu)$ be a measure space, and $u \in \mathcal{M}(\mathcal{A})$ satisfies $u^n \in \mathcal{L}(\mu)$ for all $n \geq 1$. Assume $\int u^{2n}\, d\mu = c$ and $\int u^{2n-1}\, d\mu = d$ for all $n \geq 1$, where c and d are some constants.

(a) Prove that u takes three distinct values 0, 1 and -1 μ a.e. (Hint: consider the function $u^2(u-1)^2(u+1)^2$.) (1 pt)

(b) Let $A = \{x \in X : u(x) = 1\}$ and $B = \{x \in X : u(x) = -1\}$. Prove that $c \geq d$ and that $\mu(A) = \frac{c+d}{2}$ and $\mu(B) = \frac{c-d}{2}$. (1.5 pt)

Proof (a): Using the hint, we have

$$\int u^2(u-1)^2(u+1)^2\,d\mu = \int (u^6 - 2u^4 + u^2)\,d\mu = c - 2c + c = 0.$$

By Theorem 11.2(i), we have that $u^2(u-1)^2(u+1)^2 = 0$ μ a.e. so that $u(x) \in \{0, 1, -1\}$ μ a.e.

Proof (b): Note that A and B are pairwise disjoint with $u^2 = \mathbb{I}_{A\cup B} = \mathbb{I}_A + \mathbb{I}_B$ and $u = \mathbb{I}_A - \mathbb{I}_B$ μ a.e. Thus,

$$c = \int u^2\,d\mu = \mu(A) + \mu(B),$$

and

$$d = \int u\,d\mu = \mu(A) - \mu(B).$$

From this we see that $d \leq c$ and that $\mu(A) = \frac{c+d}{2}$ and $\mu(B) = \frac{c-d}{2}$.

(4) Let $(X, \mathcal{A}, \mu_1)$ and $(Y, \mathcal{B}, \nu_1)$ be σ-finite measure spaces. Suppose $f \in \mathcal{L}^1(\mu_1)$ and $g \in \mathcal{L}^1(\nu_1)$ are **non-negative**. Define measures μ_2 on $\mathcal{A}$ and ν_2 on $\mathcal{B}$ by

$$\mu_2(A) = \int_A f\,d\mu_1 \text{ and } \nu_2(B) = \int_B g\,d\nu_1,$$

for $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

(a) Prove that if $v \in \mathcal{M}^+_{\overline{\mathbb{R}}}(\mathcal{B})$, then $\int_Y v\,d\nu_2 = \int_Y vg\,d\nu_1$. (1.5 pt)

(b) Prove that for every $D \in \mathcal{A} \otimes \mathcal{B}$ one has

$$(\mu_2 \times \nu_2)(D) = \int_D f(x)g(y)\,d(\mu_1 \times \nu_1)(x,y).$$

(1.5 pt)

Proof (a) We use the standard argument. We assume first that $v = \mathbb{I}_B$ for some $A \in \mathcal{B}$. Then

$$\int v\,d\nu_2 = \int \mathbb{I}_B\,d\nu_2 = \nu_2(B) = \int g\mathbb{I}_B\,d\nu_1 = \int vg\,d\nu_1.$$

Assume now that $v = \sum_{i=0}^n a_i\mathbb{I}_{B_i}$ is a non-negative simple function in standard form, with $B_i \in \mathcal{B}$. By linearity of the integral and the previous step, we have

$$\begin{aligned}\int v\,d\nu_2 &= \int \sum_{i=0}^n a_i\mathbb{I}_{B_i}\,d\nu_2\\ &= \sum_{i=0}^n a_i \int \mathbb{I}_{B_i}\,d\nu_2\\ &= \sum_{i=0}^n a_i \int g\mathbb{I}_{B_i}\,d\nu_1\\ &= \int g\sum_{i=0}^n a_i\mathbb{I}_{B_i}\,d\nu_1\\ &= \int gv\,d\nu_1.\end{aligned}$$

Now assume that $v \geq 0$, then by Theorem 8.8 there exists an increasing sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{E}^+(\mathcal{B})$ with $v = \sup_{n\geq 1} f_n$. By Beppo-Lévi (applied twice) and the above, we have

$$\int v\,d\nu_2 = \sup_{n\geq 1}\int f_n\,d\nu_2 = \sup_{n\geq 1}\int gf_n\,d\nu_1 = \int g\sup_{n\geq .1} f_n\,d\nu_1 = \int gv\,d\nu_1.$$

Proof (b) For $D \in \mathcal{A} \otimes \mathcal{B}$ and $y \in Y$, let $D_y = \{x \in X : (x,y) \in D\}$. Note that $D_y \in \mathcal{M}(\mathcal{A})$ and $\mathbb{I}_D(x,y) = \mathbb{I}_{D_y}(x)$. By Tonelli's Theorem and part (a), we have

$$
\begin{aligned}
(\mu_2 \times \nu_2)(D) &= \int_Y \int_X \mathbb{I}_{D_y}(x)\, d\mu_2(x)\, d\nu_2(y) \\
&= \int_Y \left(\int_X \mathbb{I}_{D_y}(x) f(x)\, d\mu_1(x) \right) d\nu_2(y) \\
&= \int_Y \left(\int_X \mathbb{I}_{D_y}(x) f(x)\, d\mu_1(x) \right) g(y)\, d\nu_1(y) \\
&= \int_Y \int_X \mathbb{I}_D(x,y) f(x) g(y)\, d\mu_1(x)\, d\nu_1(y) \\
&= \int_{X \times Y} \mathbb{I}_D(x,y) f(x) g(y)\, d(\mu_1 \times \nu_1)(x,y) \\
&= \int_D f(x) g(y)\, d(\mu_1 \times \nu_1)(x,y).
\end{aligned}
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